Malla Reddy College Engineering (Autonomous)

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Department of Information Technology

III B. TECH II SEM (A.Y.2018-19)

Lecture Notes

On

Design And Analysis Of Algorithms

DEPARTMENT OF INFORMATION TECHNOLOGY

SYLLABUS

Objectives:

- To analyze performance of algorithms.
- To choose the appropriate data structure and algorithm design method for a specified application.
- To understand how the choice of data structures and algorithm design methods impacts the performance of programs.
- To solve problems using algorithm design methods such as the greedy method, divide and conquer, dynamic programming, backtracking and branch and bound.
- Prerequisites (Subjects) Data structures, Mathematical foundations of computer science.

MODULE I:

Introduction: Algorithm, Psuedo code for expressing algorithms, Performance Analysis- Space complexity, Time complexity, Asymptotic Notation- Big oh notation, Omega notation, Theta notation and Little oh notation, Probabilistic analysis, Amortized analysis.
Divide and conquer: General method, applications-Binary search, Quick sort, Merge sort, Strassen's matrix multiplication.

MODULE II:

Searching and Traversal Techniques: Efficient non - recursive binary tree traversal algorithm, Disjoint set operations, union and find algorithms, Spanning trees, Graph traversals - Breadth first search and Depth first search, AND / OR graphs, game trees, Connected Components, Bi - connected components. Disjoint Sets- disjoint set operations, union and find algorithms, spanning trees, connected components and biconnected components.

MODULE III:

Greedy method: General method, applications - Job sequencing with deadlines, 0/1 knapsack problem, Minimum cost spanning trees, Single source shortest path problem. **Dynamic Programming:** General method, applications-Matrix chain multiplication, Optimal binary search trees, 0/1 knapsack problem, All pairs shortest path problem, Travelling sales person problem, Reliability design.

MODULE IV:

Backtracking: General method, applications-n-queen problem, sum of subsets problem, graph coloring, Hamiltonian cycles.

Branch and Bound: General method, applications - Travelling sales person problem,0/1 knapsack problem- LC Branch and Bound solution, FIFO Branch and Bound solution.

MODULE V:

NP-Hard and NP-Complete problems: Basic concepts, non deterministic algorithms, NP - Hard and NPComplete classes, Cook's theorem.

TEXT BOOKS:

1. Fundamentals of Computer Algorithms, Ellis Horowitz,Satraj Sahni and Rajasekharam,Galgotia publications pvt. Ltd.
2. Foundations of Algorithm, 4th edition, R. Neapolitan and K. Naimipour, Jones and Bartlett Learning.
3. Design and Analysis of Algorithms, P. H. Dave, H. B. Dave, Pearson Education, 2008.

REFERENCES:

1. Computer Algorithms, Introduction to Design and Analysis, 3rd Edition, Sara Baase, Allen, Van, Gelder, Pearson Education.
2. Algorithm Design: Foundations, Analysis and Internet examples, M. T. Goodrich and R. Tomassia, John Wiley and sons.
3. Fundamentals of Sequential and Parallel Algorithm, K. A. Berman and J. L. Paul, Cengage Learning.
4. Introducation to the Design and Analysis of Algorithms, A. Levitin, Pearson Education.
5. Introducation to Algorithms, 3rd Edition, T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, PHI Pvt. Ltd.
6. Design and Analysis of algorithm, Aho, Ullman and Hopcroft, Pearson Education, 2004.

Outcomes:

- Be able to analyze algorithms and improve the efficiency of algorithms.
- Apply different designing methods for development of algorithms to realistic problems, such as divide and conquer, greedy and etc. Ability to understand and estimate the performance of algorithm.

DEPARTMENT OF INFORMATION TECHNOLOGY

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MODULE I:

Introduction: Algorithm, Psuedo code for expressing algorithms, Performance Analysis-Space complexity, Time complexity, Asymptotic Notation- Big oh notation, Omega notation, Theta notation and Little oh notation, Probabilistic analysis, Amortized analysis.

Divide and conquer: General method, applications-Binary search, Quick sort, Merge sort, Strassen's matrix multiplication.

INTRODUCTION TO ALGORITHM

History of Algorithm

- The word algorithm comes from the name of a Persian author, Abu Ja'far Mohammed ibn Musa al Khowarizmi (c. 825 A.D.), who wrote a textbook on mathematics.
- He is credited with providing the step-by-step rules for adding, subtracting, multiplying, and dividing ordinary decimal numbers.
- When written in Latin, the name became Algorismus, from which algorithm is but a small step
- This word has taken on a special significance in computer science, where "algorithm" has come to refer to a method that can be used by a computer for the solution of a problem
- Between 400 and 300 B.C., the great Greek mathematician Euclid invented an algorithm
- Finding the greatest common divisor (gcd) of two positive integers.
- The gcd of X and Y is the largest integer that exactly divides both X and Y .
- Eg.,the gcd of 80 and 32 is 16.
- The Euclidian algorithm, as it is called, is considered to be the first non-trivial algorithm ever devised.

What is an Algorithm?

Algorithm is a set of steps to complete a task.

For example,

Task: to make a cup of tea. Algorithm:

· add water and milk to the kettle,
· boil it, add tea leaves,
· Add sugar, and then serve it in cup.

''a set of steps to accomplish or complete a task that is described precisely enough that a computer can run it''.

Described precisely: very difficult for a machine to know how much water, milk to be added etc. in the above tea making algorithm.

These algorithms run on computers or computational devices..For example, GPS in our smartphones, Google hangouts.

GPS uses shortest path algorithm.. **Online shopping** uses cryptography which uses RSA algorithm.

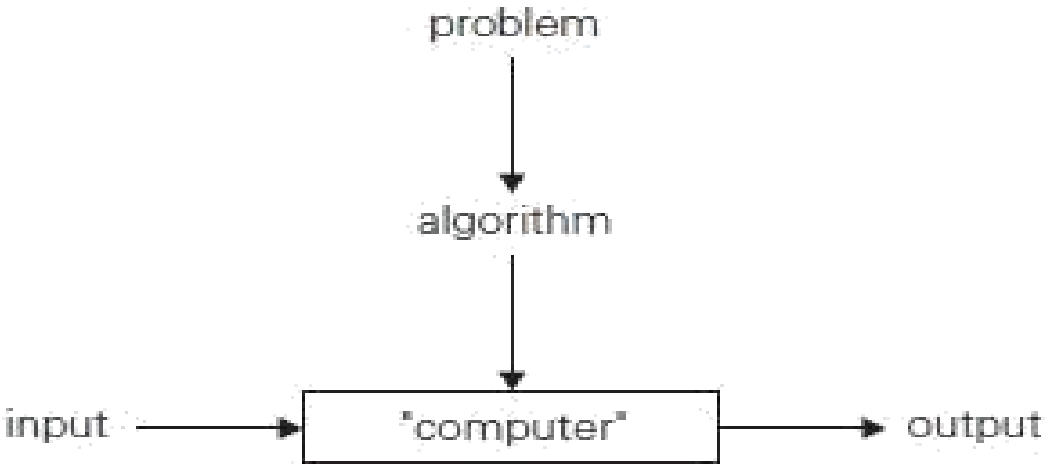
- Algorithm Definition1:

- An algorithm is a finite set of instructions that, if followed, accomplishes a particular task. In addition, all algorithms must satisfy the following criteria:
- Input. Zero or more quantities are externally supplied.
- Output. At least one quantity is produced.
- Definiteness. Each instruction is clear and unambiguous.
- Finiteness. The algorithm terminates after a finite number of steps.
- Effectiveness. Every instruction must be very basic enough and must be feasible.

- Algorithm Definition2:

- An algorithm is a sequence of unambiguous instructions for solving a problem, i.e., for obtaining a required output for any legitimate input in a finite amount of time.

- Algorithms that are definite and effective are also called *computational procedures*.
- A program is the expression of an algorithm in a programming language



- **Algorithms for Problem Solving**

The main steps for Problem Solving are:

1. Problem definition
2. Algorithm design / Algorithm specification
3. Algorithm analysis
4. Implementation
5. Testing
6. [Maintenance]

- Step1. Problem Definition
  What is the task to be accomplished?
  Ex: Calculate the average of the grades for a given student

- Step2.Algorithm Design / Specifications:
  Describe: in natural language / pseudo-code / diagrams / etc

- Step3. Algorithm analysis
  Space complexity - How much space is required
  Time complexity - How much time does it take to run the algorithm
  Computer Algorithm

An algorithm is a procedure (a finite set of well-defined instructions) for accomplishing some tasks which, given an initial state terminate in a defined end-state

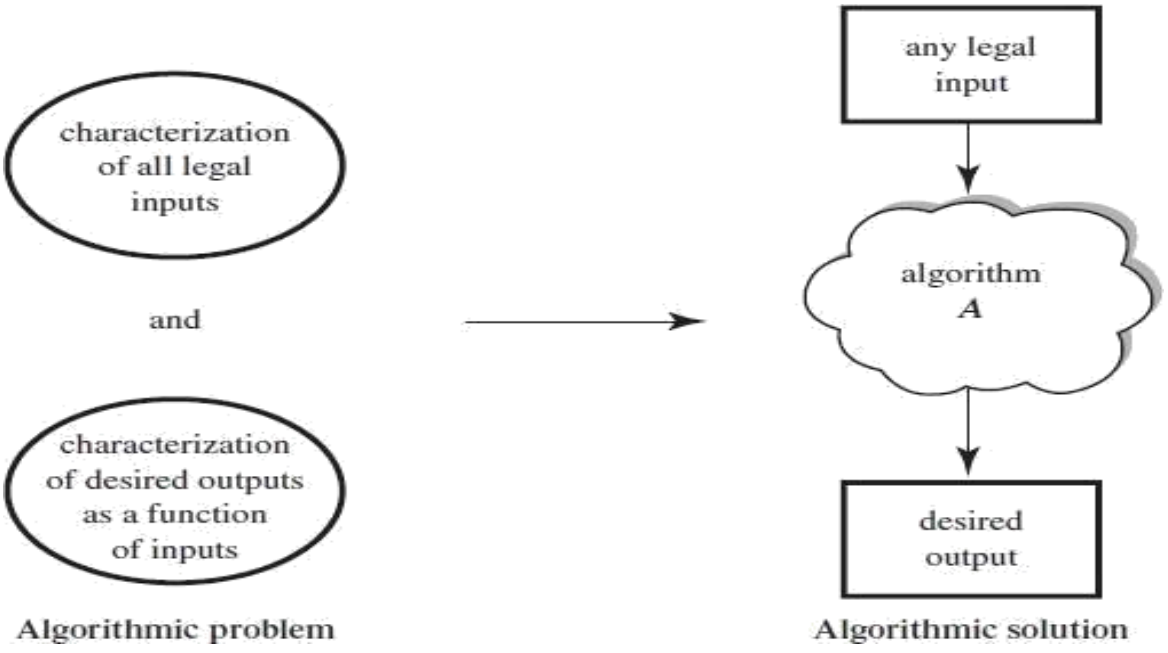
The computational complexity and efficient implementation of the algorithm are important in computing, and this depends on suitable data structures.

- Steps 4,5,6: Implementation, Testing, Maintainance
- Implementation:
  Decide on the programming language to use C, C++, Lisp, Java, Perl, Prolog, assembly, etc. , etc.

- Test, test, test
  Integrate feedback from users, fix bugs, ensure compatibility across different versions
  - Maintenance.

  Release Updates,fix bugs

Keeping illegal inputs separate is the responsibility of the algorithmic problem, while treating special classes of unusual or undesirable inputs is the responsibility of the algorithm itself.



- **4 Distinct areas of study of algorithms:**

- *How to devise algorithms*. → Techniques – Divide & Conquer, Branch and Bound , Dynamic Programming
- *How to validate algorithms*.
- Check for Algorithm that it computes the correct answer for all possible legal inputs. → algorithm validation. →
- First Phase → → → Second phase Algorithm to Program Program Proving or Program Verification Solution be stated in two forms:
- First Form: Program which is annotated by a set of assertions about the input and output variables of the program predicate calculus
- Second form: is called a specification
- 4 Distinct areas of study of algorithms (..Contd)
- *How to analyze algorithms*.
- Analysis of Algorithms or performance analysis refer to the task of determining how much computing time & storage an algorithm requires
- *How to test a program* → 2 phases
- Debugging - Debugging is the process of executing programs on sample data sets to determine whether faulty results occur and, if so, to correct them.
- Profiling or performance measurement is the process of executing a correct program on data sets and measuring the time and space it takes to compute the results

PSEUDOCODE:

- Algorithm can be represented in Text mode and Graphic mode
- Graphical representation is called Flowchart
- Text mode most often represented in close to any High level language such as C, Pascal → Pseudocode
- **Pseudocode: High-level description of an algorithm.**
- → More structured than plain English.
- → Less detailed than a program.
- → Preferred notation for describing algorithms.
- → Hides program design issues.
- **Example of Pseudocode**:

- To find the max element of an array

```
Algorithm arrayMax(A, n)
  Input array A of n integers
  Output maximum element of A
  currentMax   A[0]
  for i   1 to n − 1 do
  if A[i]   currentMax then
      currentMax   A[i]
```

return *currentMax*

- Control flow
- if … then … [else …]
- while … do …
- repeat … until …
- for … do …
- Indentation replaces braces
- Method declaration
- Algorithm *method* (*arg* [, *arg*…])
- Input …
- Output …
- Method call
- *var.method* (*arg* [, *arg*…])
- Return value
- return *expression*
- Expressions
- Assignment (equivalent to =)
- Equality testing (equivalent to ==)
- n^2 Superscripts and other mathematical formatting allowed

PERFORMANCE ANALYSIS:

- What are the Criteria for judging algorithms that have a more direct relationship to performance?
- computing time and storage requirements.

- **Performance evaluation** can be loosely divided into two major phases:
- a priori estimates and
- a posteriori testing.
- ➔ refer as *performance analysis* and *performance measurement* respectively

- The space complexity of an algorithm is the amount of memory it needs to run to completion.
- The time complexity of an algorithm is the amount of computer time it needs to run to completion.

Space Complexity:

- Space Complexity Example:
- Algorithm abc(a,b,c)

```
{
return a+b++*c+(a+b-c)/(a+b) +4.0;
}
```

→ The Space needed by each of these algorithms is seen to be the sum of the following component.

1.A fixed part that is independent of the characteristics (eg:number,size)of the inputs and outputs.

The part typically includes the instruction space (ie. Space for the code), space for simple variable and fixed-size component variables (also called aggregate) space for constants, and so on.

2. A variable part that consists of the space needed by component variables whose size is dependent on the particular problem instance being solved, the space needed by referenced variables (to the extent that is depends on instance characteristics), and the recursion stack space.

The space requirement s(p) of any algorithm p may therefore be written as, S(P)
= c+ Sp(Instance characteristics)
Where 'c' is a constant.

Example 2:

```
Algorithm sum(a,n)
{ s=0.0;
for I=1 to n do s=
s+a[I]; return s;
}
```

- The problem instances for this algorithm are characterized by n,the number of elements to be summed. The space needed d by 'n' is one word, since it is of type integer.
- The space needed by 'a'a is the space needed by variables of tyepe array of floating point numbers.
- This is atleast 'n' words, since 'a' must be large enough to hold the 'n' elements to be summed.
- So,we obtain Ssum(n)>=(n+s)
- [n for a[],one each for n,I a& s]

Time Complexity:

- The time T(p) taken by a program P is the sum of the compile time and the run time(execution time)

- The compile time does not depend on the instance characteristics. Also we may assume that a compiled program will be run several times without recompilation .This rum time is denoted by tp(instance characteristics).

- The number of steps any problem statement is assigned depends on the kind of statement.

- For example, comments à 0 steps.

Assignment statements is 1 steps.

[Which does not involve any calls to other algorithms]
Interactive statement such as for, while & repeat-untilà Control part of the statement.

We introduce a variable, count into the program statement to increment count with initial value 0.Statement to increment count by the appropriate amount are introduced into the program.

This is done so that each time a statement in the original program is executes count is incremented by the step count of that statement.

Algorithm:

```
Algorithm sum(a,n)
{
s= 0.0;
count = count+1; for I=1
to n do
{
count =count+1;
s=s+a[I];
count=count+1;
}
count=count+1;
count=count+1;  return
s;
}
```

→ If the count is zero to start with, then it will be 2n+3 on termination. So each invocation of sum execute a total of 2n+3 steps.

2. The second method to determine the step count of an algorithm is to build a table in which we list the total number of steps contributes by each statement.

→ First determine the number of steps per execution (s/e) of the statement and the total number of times (ie., frequency) each statement is executed.

→ By combining these two quantities, the total contribution of all statements, the step count for the entire algorithm is obtained.

Statement	*Steps per execution*	*Frequency*	*Total*
1. Algorithm Sum(a,n)	0	-	0
2.{	0	- 1	0
3. S=0.0;	1	+1 n 1	1
4. for I=1 to n do 5.	1	-	+1 n 1
s=s+a[I];	1		0
6. return s;	1		
7. }	0		
Total			2n+3

How to analyse an Algorithm?

Let us form an algorithm for Insertion sort (which sort a sequence of numbers).The pseudo code for the algorithm is give below.

Pseudo code for insertion Algorithm:

Identify each line of the pseudo code with symbols such as C1, C2 ..

PSeudocode for Insertion Algorithm	**Line Identification**
for j=2 to A length	C1
key=A[j]	C2
//Insert A[j] into sorted Array A[1. ..... j-1]	C3
i=j-1	C4
while i>0 & A[j]>key	C5
A[i+1]=A[i]	C6
i=i-1	C7
A[i+1]=key	C8

Let Ci be the cost of ith line. Since comment lines will not incur any cost C3=0.

Cost	**No. Of times Executed**
C1	N
C2	n-1
C3=0	n-1
C4	n-1
C5	
C6	
C7	
C8	n-1

Running time of the algorithm is:

T(n)=C1n+C2(n-1)+0(n-1)+C4(n-1)+C5( )+C6( )+C7( )+ C8(n-1)

Best case:

It occurs when Array is sorted. All t_j

values are 1.

T(n)=C1n+C2(n-1)+0 (n-1)+C4(n-1)+C5( ) +C6( )+C7( )+ C8(n-1)

=C1n+C2 (n-1) +0 (n-1) +C4 (n-1) +C5 + C8 (n-1)

= (C1+C2+C4+C5+ C8) n-(C2+C4+C5+ C8)

· Which is of the form an+b.

➔

· Linear function of n.

➔

So, linear growth.

Worst case:

It occurs when Array is reverse sorted, and tj =j

T(n)=C1n + C2(n-1)+0 (n-1)+C4(n-1)+C5( ) +C6( )+C7( ) + C8(n-1)

=C1n+C2(n-1)+C4(n-1)+C5( ) +C6( )+C7( )+ C8(n-1)

which is of the form an^2+bn+c

Quadratic function. So in worst case insertion set grows in n2. Why

we concentrate on worst-case running time?

· The worst-case running time gives a guaranteed upper bound on the running time for any input.

· For some algorithms, the worst case occurs often. For example, when searching, the worst case often occurs when the item being searched for is not present, and searches for absent items may be frequent.

· Why not analyze the average case? Because it's often about as bad as the worst case.

Order of growth:

It is described by the highest degree term of the formula for running time. (Drop lower-order terms. Ignore the constant coefficient in the leading term.)

Example: We found out that for insertion sort the worst-case running time is of the form $an^2 + bn + c$.

Drop lower-order terms. What remains is an2.Ignore constant coefficient. It results in n^2.But we cannot say that the worst-case running time T(n) equals n^2 .Rather It grows like n^2 . But it doesn't equal n^2.We say that the running time is $\Theta(n^2)$ to capture the notion that the order of growth is n^2.

We usually consider one algorithm to be more efficient than another if its worst-case running time has a smaller order of growth.

Complexity of Algorithms

The complexity of an algorithm M is the function f(n) which gives the running time and/or storage space requirement of the algorithm in terms of the size 'n' of the input data. Mostly, the storage space required by an algorithm is simply a multiple of the data size 'n'.

Complexity shall refer to the running time of the algorithm.

The function f(n), gives the running time of an algorithm, depends not only on the size 'n' of the input data but also on the particular data. The complexity function f(n) for certain cases are:

1. **Best Case** : The minimum possible value of f(n) is called the best case.

2. **Average Case** : The expected value of f(n).

3. **Worst Case** : The maximum value of f(n) for any key possible input.

ASYMPTOTIC NOTATION

➔ **Formal way notation to speak about functions and classify them**

The following notations are commonly use notations in performance analysis and used to characterize the complexity of an algorithm:

1. Big–OH (O) ,
2. Big–OMEGA (Ω),
3. Big–THETA (Θ) and
4. Little–OH (o)

Asymptotic Analysis of Algorithms:

Our approach is based on the *asymptotic complexity* measure. This means that we don't try to count the exact number of steps of a program, but how that number grows with the size of the input to the program. That gives us a measure that will work for different operating systems, compilers and CPUs. The asymptotic complexity is written using big-O notation.

· It is a way to describe the characteristics of a function in the limit.
· It describes the rate of growth of functions.
· Focus on what's important by abstracting away low-order terms and constant factors.
· It is a way to compare "sizes" of functions: O≈ ≤

$\Omega \approx \geq \Theta \approx = o \approx < \omega$
$\approx >$

Time complexity	Name	Example
O(1)	Constant	Adding an element to the front of a linked list
O(logn)	Logarithmic	Finding an element in a sorted array
O (n)	Linear	Finding an element in an unsorted array
O(nlog n)	Linear	Logarithmic Sorting n items by 'divide-and-conquer'- Mergesort
$O(n^2)$	Quadratic	Shortest path between two nodes in a graph
$O(n^3)$	Cubic	Matrix Multiplication
$O(2^n)$	Exponential	The Towers of Hanoi problem

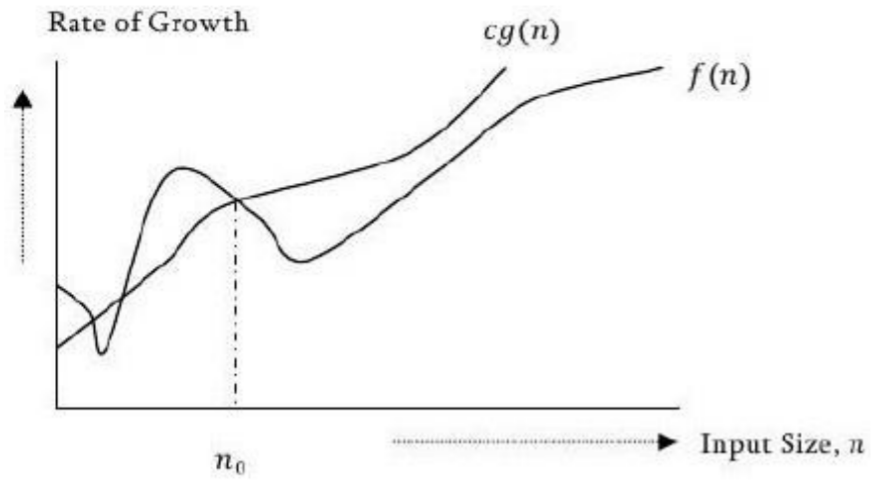
Big 'oh': the function f(n)=O(g(n)) iff there exist positive constants c and no such that f(n)<=c*g(n) for all n, n>= no.
Omega: the function f(n)=(g(n)) iff there exist positive constants c and no such that f(n) >= c*g(n) for all n, n >= no.
Theta: the function f(n)=(g(n)) iff there exist positive constants c1,c2 and no such that c1 g(n) <= f(n) <= c2 g(n) for all n, n >= no

Big-O Notation

This notation gives the tight upper bound of the given function. Generally we represent it as f(n) = O(g (11)). That means, at larger values of n, the upper bound off(n) is g(n). For example, if $f(n) = n^4 + 100n^2 + 10n + 50$ is the given algorithm, then n^4 is g(n). That means g(n) gives the maximum rate of growth for f(n) at larger values of n.

O —notation defined as O(g(n)) = {f(n): there exist positive constants c and n_o such that 0 <= f(n) <= cg(n) for all n >= n_o}. g(n) is an asymptotic tight upper bound for f(n). Our objective is to give some rate of growth g(n) which is greater than given algorithms rate of growth f(n).

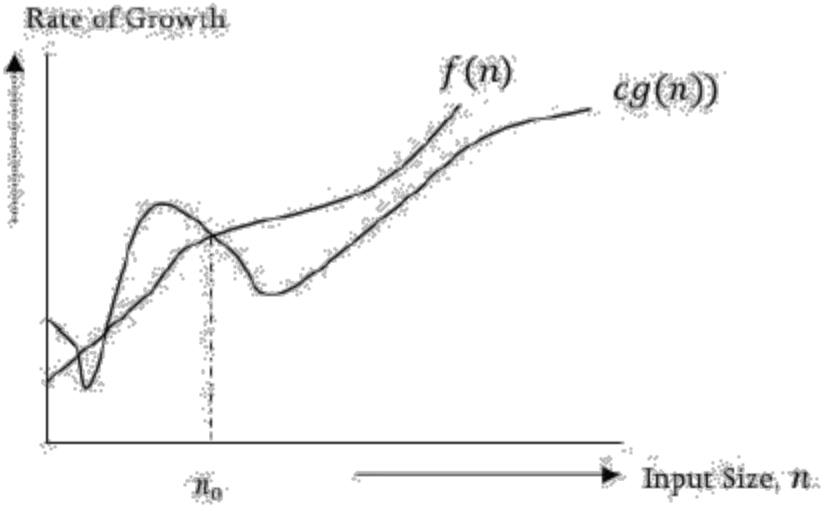
In general, we do not consider lower values of n. That means the rate of growth at lower values of n is not important. In the below figure, n_o is the point from which we consider the rate of growths for a given algorithm. Below n_o the rate of growths may be different.



Note Analyze the algorithms at larger values of n only What this means is, below no we do not care for rates of growth.

Omega— Ω notation

Similar to above discussion, this notation gives the tighter lower bound of the given algorithm and we represent it as f(n) = Ω (g(n)). That means, at larger values of n, the tighter lower bound of f(n) is g
For example, if f(n) = $100n^2$ + 10n + 50, g(n) is Ω (n^2).
The . Ω. notation as be defined as Ω (g (n)) = {f(n): there exist positive constants c and n_o such that 0 <= cg (n) <= f(n) for all n >= n_o}. g(n) is an asymptotic lower bound for f(n). Ω (g (n)) is the set of functions with smaller or same order of growth as f(n).

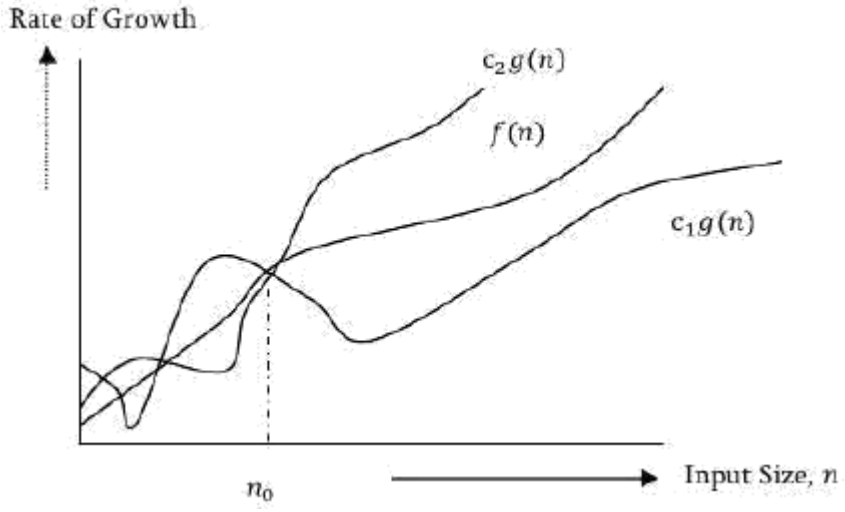


Theta- Θ notation

This notation decides whether the upper and lower bounds of a given function are same or not. The average running time of algorithm is always between lower bound and upper bound.

If the upper bound (O) and lower bound (Ω) gives the same result then Θ notation will also have the same rate of growth. As an example, let us assume that f(n) = 10n + n is the expression. Then, its tight upper bound g(n) is O(n). The rate of growth in best case is g (n) = 0(n). In this case, rate of growths in best case and worst are same. As a result, the average case will also be same.

None: For a given function (algorithm), if the rate of growths (bounds) for O and Ω are not same then the rate of growth Θ case may not be same.



Now consider the definition of Θ notation It is defined as Θ (g(n)) = {f(71): there exist positive constants C1, C2 and no such that O<=5 c_1g(n) <= f(n) <= c_2g(n) for all n >= n_0}. g(n) is an asymptotic tight bound for f(n). Θ (g(n)) is the set of functions with the same order of growth as g(n).

Important Notes

For analysis (best case, worst case and average) we try to give upper bound (O) and lower bound (Ω) and average running time (Θ). From the above examples, it should also be clear that, for a given function (algorithm) getting upper bound (O) and lower bound (Ω) and average running time (Θ) may not be possible always.
For example, if we are discussing the best case of an algorithm, then we try to give upper bound (O) and lower bound (Ω) and average running time (Θ).
In the remaining chapters we generally concentrate on upper bound (O) because knowing lower bound (Ω) of an algorithm is of no practical importance and we use 9 notation if upper bound (O) and lower bound (Ω) are same.

Little Oh Notation

The little Oh is denoted as o. It is defined as : Let, f(n} and g(n} be the non negative functions then

such that f(n}= o(g{n)} i.e f of n is little Oh of g of n.

f(n) = o(g(n)) if and only if f'(n) = o(g(n)) and f(n) != Θ {g(n))

PROBABILISTIC ANALYSIS

Probabilistic analysis is the use of probability in the analysis of problems.

In order to perform a probabilistic analysis, we must use knowledge of, or make assumptions about, the distribution of the inputs. Then we analyze our algorithm, computing an average- case running time, where we take the average over the distribution of the possible inputs.

Basics of Probability Theory

Probability theory has the goal of characterizing the outcomes of natural or conceptual "experiments." Examples of such experiments include tossing a coin ten times, rolling a die three times, playing a lottery, gambling, picking a ball from an urn containing white and red balls, and so on

Each possible outcome of an experiment is called a sample point and the set of all possible outcomes is known as the sample space S. In this text we assume that S is finite (such a sample space is called a discrete sample space). An event E is a subset of the sample space S. If the sample space consists of n sample points, then there are 2^n possible events.

Definition- Probability: The probability of an event E is defined to be where S — is the sample space.

Then the ***indicator random variable*** I {A} associated with event A is defined as I {A}

= 1 if A occurs ;
0 if A does not occur

The probability of event E is denoted as Prob. [E] The complement of E, denoted E, is defined to be S - E. If E1 and E2 are two events, the probability of E1 or E2 or both happening is denoted as Prob.[E1 U E2]. The probability of both E1 and E2 occurring at the same time is denoted as Prob.[E1 0 E2]. The corresponding event is E1 0 E2.

Theorem 1.5

1. Prob.[E] = 1 - Prob.[E].
2. Prob.[E1 U E2] = Prob.[E1] + Prob.[E2] - Prob.[E1 ∩ E2]
   <= Prob.[E1] + Prob.[E2]

Expected value of a random variable

The simplest and most useful summary of the distribution of a random variable is the average" of the values it takes on. The ***expected value*** (or, synonymously, ***expectation*** or ***mean***) of a discrete random variable X is

E[X] =

which is well defined if the sum is finite or converges absolutely.

Consider a game in which you flip two fair coins. You earn $3 for each head but lose $2 for each tail. The expected value of the random variable X representing

your earnings is

E[X] = 6.Pr{2H's} + 1.Pr{1H,1T} – 4 Pr{2T's}

= 6(1/4)+1(1/2)-4(1/4)
=1

Any one of these first i candidates is equally likely to be the best-qualified so far. Candidate i has a probability of 1/i of being better qualified than candidates 1 through i -1 and thus a probability of 1/i of being hired.

E[Xi]= 1/i

So,
E[X] = E[ ]

=
=

AMORTIZED ANALYSIS

In an ***amortized analysis***, we average the time required to perform a sequence of datastructure operations over all the operations performed. With amortized analysis, we can show that the average cost of an operation is small, if we average over a sequence of operations, even though a single operation within the sequence might be expensive. Amortized analysis differs from average-case analysis in that probability is not involved; an amortized analysis guarantees the *average performance of each operation in the worst case*.

Three most common techniques used in amortized analysis:

1. **Aggregate Analysis -** in which we determine an upper bound T(n) on the total cost of a sequence of n operations. The average cost per operation is then T(n)/n. We take the average cost as the amortized cost of each operation

2. **Accounting method** – When there is more than one type of operation, each type of operation may have a different amortized cost. The accounting method overcharges some operations early in the sequence, storing the overcharge as "prepaid credit" on specific objects in the data structure. Later in the sequence, the credit pays for operations that are charged less than they actually cost.

3. **Potential method** - The potential method maintains the credit as the "potential energy" of the data structure as a whole instead of associating the credit with individual objects within the data structure. The potential method, which is like the accounting method in that we determine the amortized cost of each operation and may overcharge operations early on to compensate for undercharges later

DIVIDE AND CONQUER

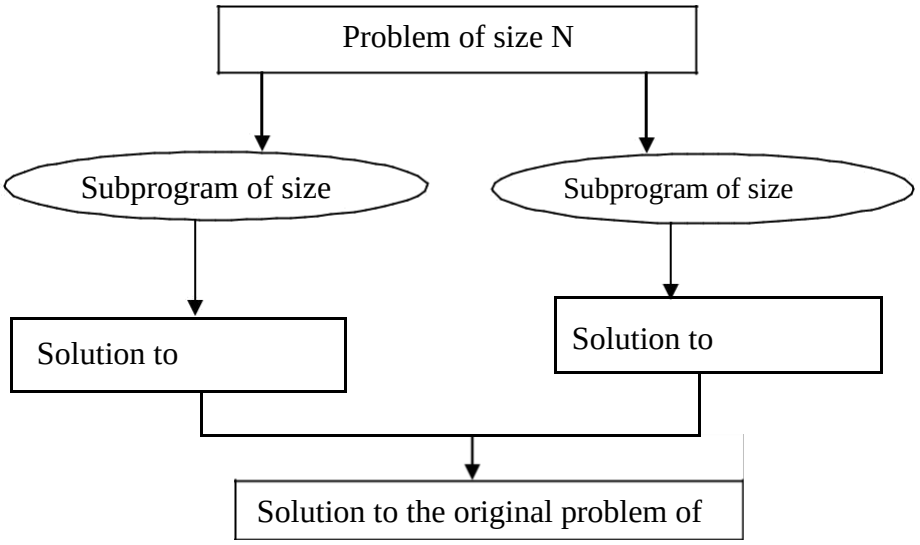
General Method

In divide and conquer method, a given problem is,

i) Divided into smaller subproblems.
ii) These subproblems are solved independently.
iii) Combining all the solutions of subproblems into a solution of the whole.

If the subproblems are large enough then divide and conquer is reapplied.
The generated subproblems are usually of some type as the original problem.

Hence recurssive algorithms are used in divide and conquer strategy.



Pseudo code Representation of Divide and conquer rule for problem "P"

```
Algorithm DAndC(P)
{
if small(P) then return S(P)
else{
divide P into smaller instances P1,P2,P3…Pk;
apply DAndC to each of these subprograms; // means DAndC(P1), DAndC(P2)…..
DAndC(Pk)
return combine(DAndC(P1), DAndC(P2)….. DAndC(Pk));
}
}
```

→

//P Problem

//Here small(P) → **Boolean value function. If it is true, then the function S is //invoked**

Time Complexity of DAndC algorithm:

```
T(n) = T(1) if n=1
       aT(n/b)+f(n) if n>1
```

→ a,b contants.

This is called the **general divide and-conquer recurrence.**

Example for GENERAL METHOD:

As an example, let us consider the problem of computing the sum of n numbers $a_0, \ldots a_{n-1}$.
If n > 1, we can divide the problem into two instances of the same problem. They are sum of the first | n/2|numbers
Compute the sum of the 1st [n/2] numbers, and then compute the sum of another n/2 numbers.
Combine the answers of two n/2 numbers sum.
i.e.,

$a_0 + \ldots + a_{n-1} = (a_0 + \ldots + a_{n/2}) + (a_{n/2} + \ldots + a_{n-1})$

Assuming that size n is a power of b, to simplify our analysis, we get the following recurrence for the running time T(n).

T(n)=aT(n/b)+f(n)

This is called the general **divide and-conquer recurrence**.

f(n) → is a function that accounts for the time spent on dividing the problem into smaller ones and on combining their solutions. (For the summation example, a = b = 2 and f (n) = 1.

Advantages of DAndC:

The time spent on executing the problem using DAndC is smaller than other method. This technique is ideally suited for parallel computation.
This approach provides an efficient algorithm in computer science.

Master Theorem for Divide and Conquer

In all efficient divide and conquer algorithms we will divide the problem into subproblems, each of which is some part of the original problem, and then perform some additional work to compute the final answer. As an example, if we consider merge sort [for details, refer Sorting chapter], it operates on two problems, each of which is half the size of the original, and then uses O(n) additional work for merging. This gives the running time equation:

T(n) = 2T( )+ O(n)

The following theorem can be used to determine the running time of divide and conquer algorithms. For a given program or algorithm, first we try to find the recurrence relation for the problem. If the recurrence is of below form then we directly give the answer without fully solving it.

If the reccurrence is of the form T(n) = aT( ) + Θ (nklog^{p}n), where a >= 1, b > 1, k >= O and p is a real number, then we can directly give the answer as:

1) If a > b^{k}, then T(n) = Θ ( )
2) If a = b^{k}
   a. If p > -1, then T(n) = Θ ( )
   b. If p = -1, then T(n) = Θ ( )
   c. If p < -1, then T(n) = Θ (s)
3) If a < b^{k}
   a. If p >= 0, then T(n) = Θ (n^{k}logp^{n})
   b. If p < 0, then T(n) = 0(n^{k})

Applications of Divide and conquer rule or algorithm:

- Binary search,
- Quick sort,
- Merge sort,
- Strassen's matrix multiplication.

Binary search or Half-interval search algorithm:

1. This algorithm finds the position of a specified input value (the search "key") within an array sorted by key value.
2. In each step, the algorithm compares the search key value with the key value of the middle element of the array.
3. If the keys match, then a matching element has been found and its index, or position, is returned.
4. Otherwise, if the search key is less than the middle element's key, then the algorithm repeats its action on the sub-array to the **left** of the middle element or, if the search key is greater, then the algorithm repeats on sub array to the **right** of the middle element.
5. If the search element is less than the minimum position element or greater than the maximum position element then this algorithm returns not found.

Binary search algorithm by using recursive methodology:

Program for binary search (recursive)	Algorithm for binary search (recursive)
int binary_search(int A[], int key, int imin, int imax)	**Algorithm binary_search(A, key, imin, imax)**

{
if (imax < imin)
return array is empty;
if(key<imin || K>imax)
return element not in array list else
{
int imid = (imin +imax)/2; if
(A[imid] > key)
return binary_search(A, key, imin, imid-1); else
if (A[imid] < key)
return binary_search(A, key, imid+1, imax); else
return imid;
}
}
	{
	if (imax < imin) then
	return "array is empty";
	if(key<imin || K>imax) then
	return "element not in array list"
	else
	{
	mid = (imin +imax)/2; if
	(A[imid] > key) then
	urn binary_search(A, key, imin, imid-1); else if (A[imid] < key) then
	urn binary_search(A, key, imid+1, imax); else
	return imid;
	}
	}

Time Complexity:

Data structure:- Array

For successful search	Unsuccessful search
Worst case → O(log n) or θ(log n) Average case → O(log n) or θ(log n) Best case → O(1) or θ(1)	θ(log n):- for all cases.

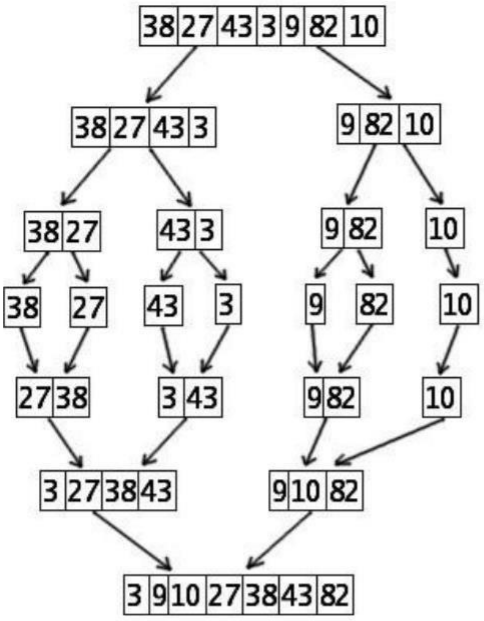
Binary search algorithm by using iterative methodology:

Binary search program by using iterative methodology:	Binary search algorithm by using iterative methodology:

int binary_search(int A[], int key, int imin, int imax)
{
while (imax >= imin)
{
int imid = midpoint(imin, imax);
if(A[imid] == key)
return imid;
else if (A[imid] < key) imin
= imid + 1;
else
imax = imid - 1;
}
}
	Algorithm binary_search(A, key, imin, imax)
	{
	While < (imax >= imin)> do
	{
	int imid = midpoint(imin, imax);
	if(A[imid] == key)
	return imid;
	else if (A[imid] < key) imin
	= imid + 1;
	else
	imax = imid - 1;
	}
	}

Merge Sort:

The merge sort splits the list to be sorted into two equal halves, and places them in separate arrays. This sorting method is an example of the DIVIDE-AND-CONQUER paradigm i.e. it breaks the data into two halves and then sorts the two half data sets recursively, and finally merges them to obtain the complete sorted list. The merge sort is a comparison sort and has an algorithmic complexity of $O(n \log n)$. Elementary implementations of the merge sort make use of two arrays - one for each half of the data set. The following image depicts the complete procedure of merge sort.



Advantages of Merge Sort:

1. Marginally faster than the heap sort for larger sets
2. Merge Sort always does lesser number of comparisons than Quick Sort. Worst case for merge sort does about 39% less comparisons against quick sort's average case.
3. Merge sort is often the best choice for sorting a linked list because the slow random- access performance of a linked list makes some other algorithms (such as quick sort) perform poorly, and others (such as heap sort) completely impossible.

Program for Merge sort:

```
#include<stdio.h>
#include<conio.h>
int n;
void main(){
int i,low,high,z,y;
int a[10];
void mergesort(int a[10],int low,int high);
void display(int a[10]);
clrscr();
printf("\n \t\t mergesort \n");
printf("\n enter the length of the list:");
scanf("%d",&n);
printf("\n enter the list elements");
for(i=0;i<n;i++)
scanf("%d",&a[i]);
low=0;
high=n-1;
mergesort(a,low,high);
display(a);
getch();
}
void mergesort(int a[10],int low, int high)
```

```
{
int mid;
void combine(int a[10],int low, int mid, int high);
if(low<high)
{
mid=(low+high)/2;
mergesort(a,low,mid);
mergesort(a,mid+1,high);
combine(a,low,mid,high);
}
}
void combine(int a[10], int low, int mid, int
high){ int i,j,k;
int temp[10];
k=low; i=low;
j=mid+1;
while(i<=mid&&j<=high){
if(a[i]<=a[j])
{
temp[k]=a[i]; i++;
k++;
}
else
{
temp[k]=a[j]; j++;
k++;
}
}
while(i<=mid){
temp[k]=a[i]; i++;
k++;
}

while(j<=high){
temp[k]=a[j]; j++;
k++;
}
for(k=low;k<=high;k++)
a[k]=temp[k];
}
void display(int a[10]){ int i;
printf("\n \n the sorted array is \n");
for(i=0;i<n;i++)
printf("%d \t",a[i]);}
```

Algorithm for Merge sort:

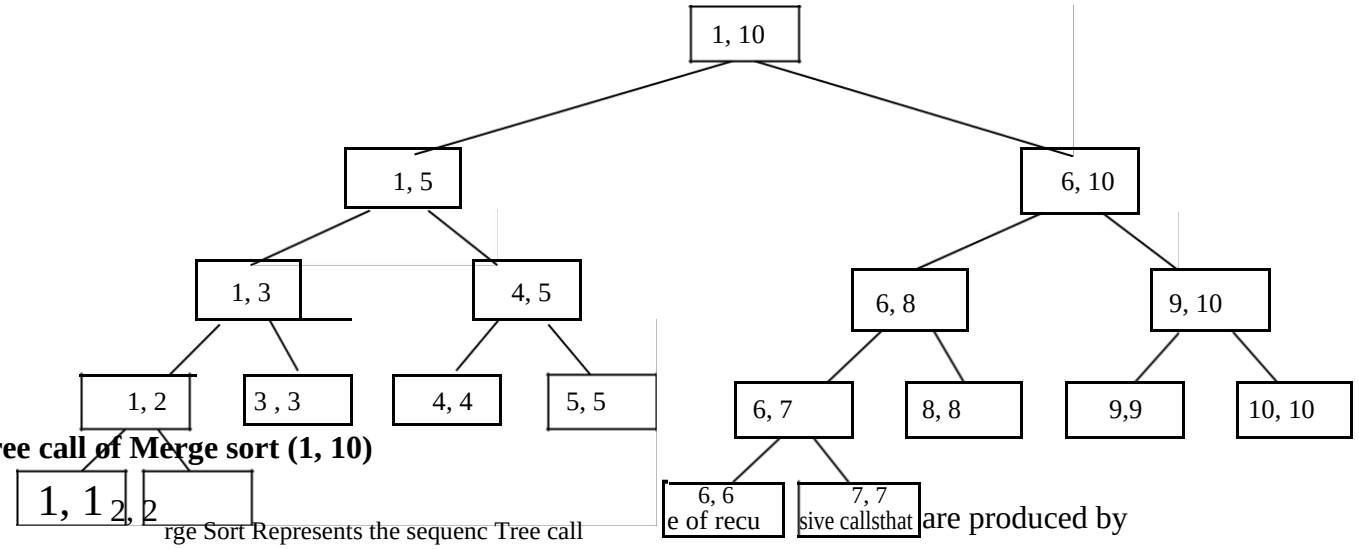
```
Algorithm mergesort(low, high)
{
if(low<high) then
{
mid=(low+high)/2;
mergesort(low,mid);
mergesort(mid+1,high); //Solve the sub-problems
Merge(low,mid,high); // Combine the solution
}
}
void Merge(low, mid,high){
k=low;
i=low;
j=mid+1;
while(i<=mid&&j<=high) do{
if(a[i]<=a[j]) then
{
temp[k]=a[i];
i++;
k++;
}
else
{
temp[k]=a[j];
j++;
k++;
}
}
while(i<=mid) do{
temp[k]=a[i];
i++;
k++;
}

while(j<=high) do{
temp[k]=a[j];
j++;
k++;
}
For k=low to high do
a[k]=temp[k];
}
For k:=low to high do a[k]=temp[k];
}
```

// Dividing Problem into Sub-problems and this "mid" is for finding where to split the set.

Tree call of Merge sort

Consider a example: (From text book)

A[1:10]={310,285,179,652,351,423,861,254,450,520}



Tree call of Merge sort (1, 10)

1, 1 2, 2 rge Sort Represents the sequenc Tree call of Me e of recu sive callsthat are produced by merge sort.

"Once observe the explained notes in class room"

Computing Time for Merge sort:

The time for the merging operation in proportional to n, then computing time for merge sort is described by using recurrence relation.

T(n)= a	**if n=1;**
2T(n/2)+ cn	**if n>1**

Here c, a $\longrightarrow$ Constants.

If n is power of 2, $n=2^k$

Form recurrence relation T(n)=

2T(n/2) + cn

2[2T(n/4)+cn/2] + cn

4T(n/4)+2cn

2^2 T(n/4)+2cn

2^3 T(n/8)+3cn

2^4 T(n/16)+4cn 2^k

T(1)+kcn an+cn(log

n)

By representing it by in the form of Asymptotic notation O is

T(n)=O(nlog n)

Quick Sort

Quick Sort is an algorithm based on the DIVIDE-AND-CONQUER paradigm that selects a pivot element and reorders the given list in such a way that all elements smaller to it are on one side and those bigger than it are on the other. Then the sub lists are recursively sorted until the list gets completely sorted. The time complexity of this algorithm is O (n log n).

- Auxiliary space used in the average case for implementing recursive function calls is O (log n) and hence proves to be a bit space costly, especially when it comes to large data sets.
- Its worst case has a time complexity of O (n ^{2}) which can prove very fatal for large data sets.

Competitive sorting algorithms

Quick sort program

```
#include<stdio.h>
#include<conio.h>
int n,j,i;
void main(){
int i,low,high,z,y; int
a[10],kk;
void quick(int a[10],int low,int high);
int n;
clrscr();
printf("\n \t\t mergesort \n");
printf("\n enter the length of the list:");
scanf("%d",&n);
printf("\n enter the list elements");
for(i=0;i<n;i++)
scanf("%d",&a[i]);
low=0;
high=n-1;
quick(a,low,high); printf("\n
sorted array is:");
for(i=0;i<n;i++)
printf(" %d",a[i]);
getch();
}

int partition(int a[10], int low, int high){
int i=low,j=high;
int temp;
int mid=(low+high)/2;
int pivot=a[mid];
while(i<=j)
{
while(a[i]<=pivot)
i++;
```

```
while(a[j]>pivot)
j--;
if(i<=j){
        temp=a[i];
        a[i]=a[j];
        a[j]=temp;
        i++;
        j--;
}}
return j;
}
void quick(int a[10],int low, int high)
{
int m=partition(a,low,high);
if(low<m)
quick(a,low,m);
if(m+1<high)
quick(a,m+1,high);
}
```

Algorithm for Quick sort

```
Algorithm quickSort (a, low, high) {
If(high>low) then{
m=partition(a,low,high);
if(low<m) then quick(a,low,m);
if(m+1<high) then quick(a,m+1,high);
}}

Algorithm partition(a, low, high){
i=low,j=high;
mid=(low+high)/2;
pivot=a[mid];
while(i<=j) do { while(a[i]<=pivot)
i++;
 hile(a[j]>pivot) j--;
if(i<=j){ temp=a[i];
a[i]=a[j]; a[j]=temp;
i++;
j--;
}}
return j;
}
```

Name	Time Complexity			Space Complexity
	Best case	verage Case	Worst Case	
Bubble	O(n)	-	$O(n^2)$	O(n)
Insertion	O(n)	$O(n^2)$	$O(n^2)$	O(n)
Selection	$O(n^2)$	$O(n^2)$	$O(n^2)$	O(n)

Quick	O(log n)	O(n log n)	$O(n^2)$	O(n + log n)
Merge	O(n log n)	O(n log n)	O(n log n)	O(2n)
Heap	O(n log n)	O(n log n)	O(n log n)	O(n)

Comparison between Merge and Quick Sort:

- Both follows Divide and Conquer rule.
- Statistically both merge sort and quick sort have the same average case time i.e., O(n log n).
- Merge Sort Requires additional memory. The pros of merge sort are: it is a stable sort, and there is no worst case (means average case and worst case time complexity is same).
- Quick sort is often implemented in place thus saving the performance and memory by not creating extra storage space.
- But in Quick sort, the performance falls on already sorted/almost sorted list if the pivot is not randomized. Thus why the worst case time is $O(n^2)$.

Randomized Sorting Algorithm: (Random quick sort)

- While sorting the array a[p:q] instead of picking a[m], pick a random element (from among a[p], a[p+1], a[p+2]--- a[q]) as the partition elements.
- The resultant randomized algorithm works on any input and runs in an expected O(n log n) times.

Algorithm for Random Quick sort

```
Algorithm RquickSort (a, p, q) { If(high>low)
then{
If((q-p)>5) then
nterchange(a, Random() mod (q-p+1)+p, p);
m=partition(a,p, q+1);
quick(a, p, m-1);
quick(a,m+1,q);
}}
```

Strassen's Matrix Multiplication:

Let A and B be two n×n Matrices. The product matrix C=AB is also a n×n matrix whose i, j^{th} element is formed by taking elements in the i^{th} row of A and j^{th} column of B and multiplying them to get

C(i, j)=

Here $1 \leq i \& j \leq n$ means i and j are in between 1 and n.

To compute C(i, j) using this formula, we need n multiplications.

The divide and conquer strategy suggests another way to compute the product of two n×n matrices.

For Simplicity assume n is a power of 2 that is $n=2^k$

If n is not power of two then enough rows and columns of zeros can be added to both A and B, so that resulting dimensions are a power of two.

Let A and B be two n×n Matrices. Imagine that A & B are each partitioned into four square sub matrices. Each sub matrix having dimensions n/2×n/2.
The product of AB can be computed by using previous formula.
If AB is product of 2×2 matrices then

=

$C_{11}=A_{11}B_{11}+A_{12}B_{21}$
$C_{12}=A_{11}B_{12}+A_{12}B_{22}$
$C_{21}=A_{21}B_{11}+A_{22}B_{21}$ $C_{22}=A_{21}B_{12}+A_{22}B_{22}$

Here 8 multiplications and 4 additions are performed.
Note that Matrix Multiplication are more Expensive than matrix addition and subtraction.

$T(n)= b$ if $n\leq 2$;
$8T(n/2)+ cn^2$ if $n>2$

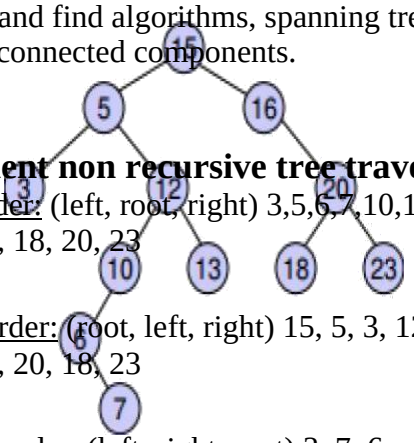
Volker strassen has discovered a way to compute the $C_{i,j}$ of above using 7 multiplications and 18 additions or subtractions.
For this first compute 7 n/2×n/2 matrices P, Q, R, S, T, U & V
P=(A11+A22)(B11+B22)
Q=(A21+A22)B11
R=A11(B12-B22) S=A22(B21-B11) T=(A11+A12)B22
U=(A21-A11)(B11+B12)
V=(A12-A22)(B21+B22)

C11=P+S-T+V
C12=R+T C21=Q+S
C22=P+R-Q+U

$T(n)= b$ if $n\leq 2$;
$7T(n/2)+ cn^2$ if $n>2$

MODULE II:

Searching and Traversal Techniques: Efficient non - recursive binary tree traversal algorithm, Disjoint set operations, union and find algorithms, Spanning trees, Graph traversals - Breadth first search and Depth first search, AND / OR graphs, game trees, Connected Components, Bi - connected components. Disjoint Sets- disjoint set operations,
union and find algorithms, spanning trees, connected components
and biconnected components.



Efficient non recursive tree traversal algorithms

in-order: (left, root, right) 3,5,6,7,10,12,13
15, 16, 18, 20, 23

pre-order: (root, left, right) 15, 5, 3, 12, 10, 6, 7,
13, 16, 20, 18, 23

post-order: (left, right, root) 3, 7, 6,
10, 13, 12, 5,
18, 23,20,16, 65

Non recursive Inorder traversal algorithm

1. Start fiom the root. let's it is current.
2. Ifcurrent is not NULL. push the node on to stack.
3. Move to left child of current and go to step 2.
4. Ifcurrent is NULL, and stack is not empty, pop node from the stack.
5. Print the node value and change current to right child of current.
6. Go to step 2.

So we go on traversing all left node. as we visit the node. we will put that node into stack.remember need to visit parent after the child and as We will encounter parent first when start from root. it's case for LIFO :) and hence the stack). Once we reach NULL node. we will take the node at the top of the stack. last node which we visited. Print it.
Check if there is right child to that node. If yes. move right child to stack and again start traversing left child node and put them on to stack. Once we have traversed all node. our stack will be empty.

Non recursive postorder traversal algorithm Left

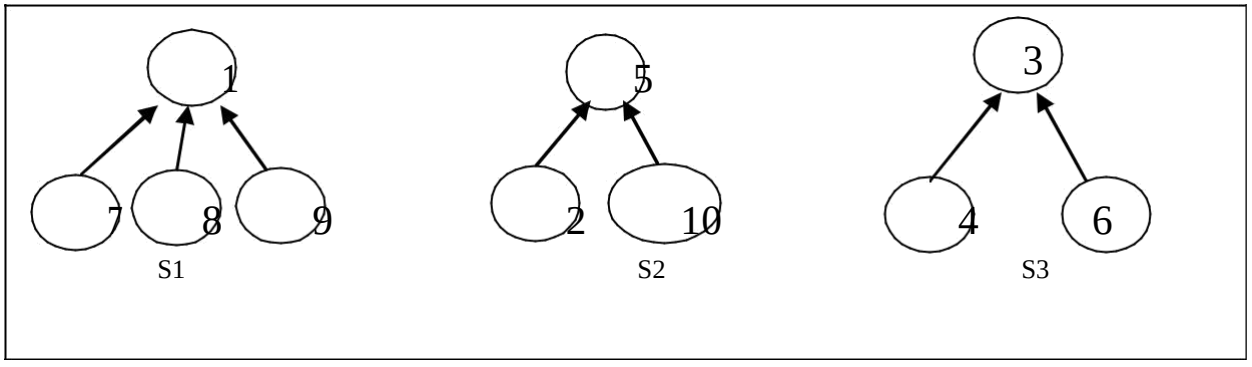
node. right node and last parent
node. 1.1 Create an empty stack
Do Following while root is not NULL
a) Push root's right child and then root to stack.

b) Set root as root's left child.
Pop an item from stack and set it as root.
a) If the popped item has a right child and the right child is at top of stack, then remove the right child from stack, push the root back and set root as root's right child.
Ia) Else print root's data and set root as NULL.
Repeat steps 2.1 and 2.2 while stack is not empty.

Disjoint Sets: If S_i and S_j, $i \neq j$ are two sets, then there is no element that is in both S_i and S_j..
For example: n=10 elements can be partitioned into three disjoint sets,

S_1= {1, 7, 8, 9}
S_2= {2, 5, 10}
S_3= {3, 4, 6}

Tree representation of sets:

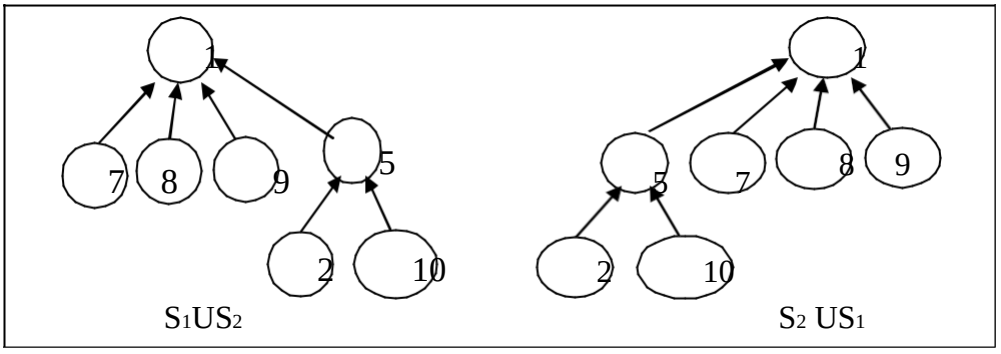


- **Disjoint set Operations**:
- Disjoint set Union

Find(i)

nion: Means Combination of two disjoint sets elements. Form above example $S_1 \cup S_2$

S_1US_2
={1,7,8,9,5,2,10 }
For S_1 U S_2 tree representation, simply make one of the tree is a subtree of the other.



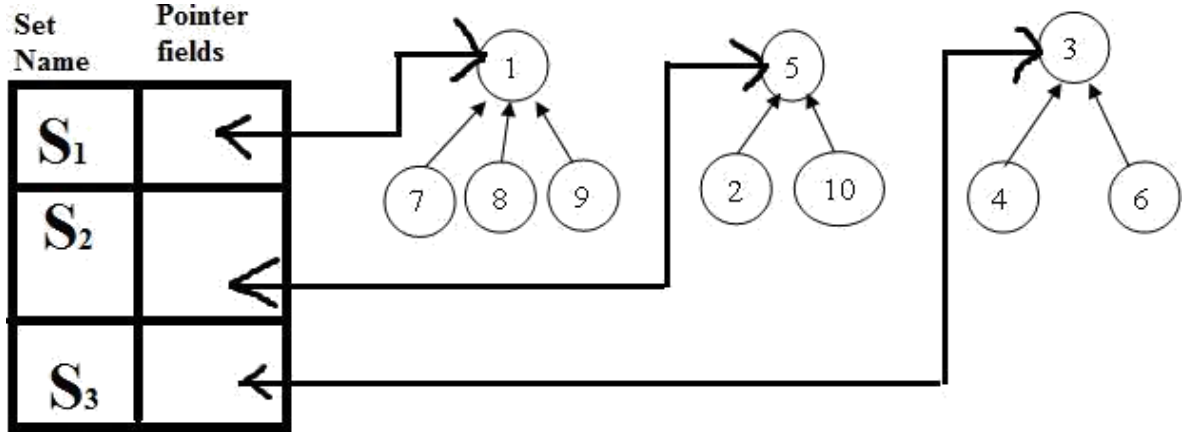
Find: Given element i, find the set containing i.
Form above example:

Find(4) → S_3

Find(1) → S_1 Find(10) → S_2

Data representation of sets:

Tress can be accomplished easily if, with each set name, we keep a pointer to the root of the tree representing that set.



For presenting the union and find algorithms, we ignore the set names and identify sets just by the roots of the trees representing them.

For example: if we determine that element 'i' is in a tree with root 'j' has a pointer to entry 'k' in the set name table, then the set name is just **name[k]**

For unite (**adding or combine**) to a particular set we use FindPointer function.

ample: If you wish to unite to S_i and S_j then we wish to unite the tree with roots FindPointer (S_i) and FindPointer (S_j)

inter → is a function that takes a set name and determines the root of the tree that represents it.

For determining operations:

Find(i) → 1^{St} determine the root of the tree and find its pointer to entry in setname table.

Union(i, j) → Means union of two trees whose roots are i and j.

If set contains numbers 1 through n, we represents tree node **P[1:n]**.

n → Maximum number of elements. Each

i	1	2	3	4	5	6	7	8	9	10
P	-1	5	-1	3	-1	3	1	1	1	5

Find(i) by following the indices, starting at i until we reach a node with parent value -1.

Example: Find(6) start at 6 and then moves to 6's parent. Since P[3] is negative, we reached the root.

Algorithm for finding Union(i, j):	Algorithm for find(i)

Algorithm Simple union(i, j)
{
P[i]:=j; // Accomplishes the union
}
	Algorithm SimpleFind(i)
	{
	While(P[i]≥0) do i:=P[i];
	return i;
	}

If n numbers of roots are there then the above algorithms are not useful for union and find. For union of n trees → Union(1,2), Union(2,3), Union(3,4),…..Union(n-1,n).

For Find i in n trees → Find(1), Find(2),….Find(n).

Time taken for the union (simple union) is → O(1) (constant).

For the n-1 unions → O(n).

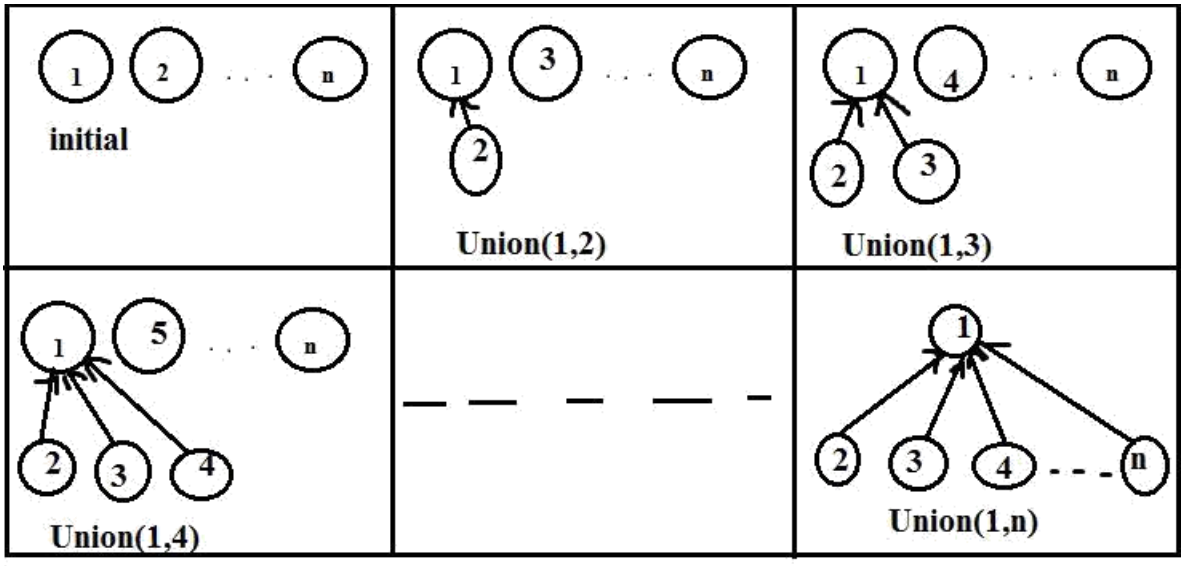
Time taken for the find for an element at level i of a tree is → O(i).

For n finds → $O(n^2)$.

To improve the performance of our union and find algorithms by avoiding the creation of degenerate trees. For this we use a weighting rule for union(i, j)

Weighting rule for Union(i, j):

If the number of nodes in the tree with root 'i' is less than the tree with root 'j', then make 'j' the parent of 'i'; otherwise make 'i' the parent of 'j'.



Tree obtained using the weighting rule

Algorithm for weightedUnion(i, j)

```
Algorithm WeightedUnion(i,j)
//Union sets with roots i and j, i≠j
// The weighting rule, p[i]= -count[i] and p[j]= -count[j].
{
temp := p[i]+p[j];
if (p[i]>p[j]) then
{ // i has fewer
nodes. P[i]:=j;
P[j]:=temp;
}
else
{ // j has fewer or equal
nodes. P[j] := i;
P[i] := temp;
}
}
```

For implementing the weighting rule, we need to know how many nodes there are in every tree.

For this we maintain a count field in the root of every tree. i → root node

count[i] → number of nodes in the tree.

Time required for this above algorithm is O(1) + time for remaining unchanged is determined by using **Lemma.**

a:- Let T be a tree with **m** nodes created as a result of a sequence of unions each performed using WeightedUnion. The height of T is no greater than $\lfloor \log_2 m \rfloor + 1$.

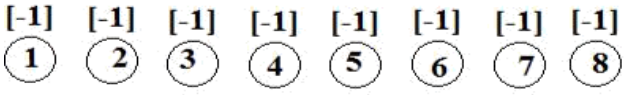
le: If ‘j’ is a node on the path from ‘i’ to its root and p[i]≠root[i], then set p[j] to root[i].

Algorithm for Collapsing find.

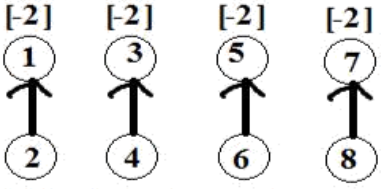
Algorithm CollapsingFind(i)
//Find the root of the tree containing element i.
//collapsing rule to collapse all nodes form i to the root.
{
r;=i;
while(p[r]>0) do r := p[r]; //Find the root. While(i
≠ r) do // Collapse nodes from i to root r.
{
s:=p[i];
p[i]:=r;
i:=s;
}
return r;
}

Collapsing find algorithm is used to perform find operation on the tree created by WeightedUnion.

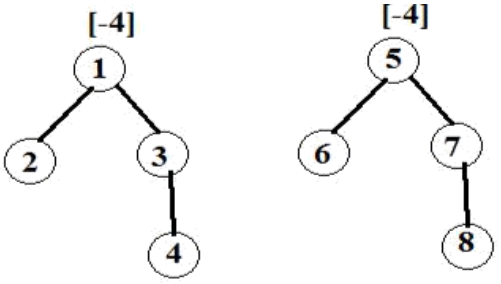
For example: Tree created by using WeightedUnion



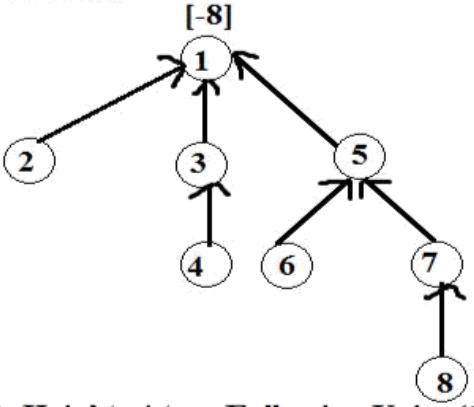
(a) initial height -1 tree



(b) Height -2 trees follwoing Union(1,2),(3,4),(5,6),(7,8)



(c) Height -3 trees following Union (1,3) and (5,7)



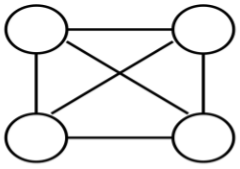
(d) Height -4 tree Following Union(1,5)

Now process the following eight finds: Find(8), Find(8),… ..... Find(8)
If SimpleFind is used, each Find(8) requires going up three parent link fields for a total of 24 moves to process all eight finds.

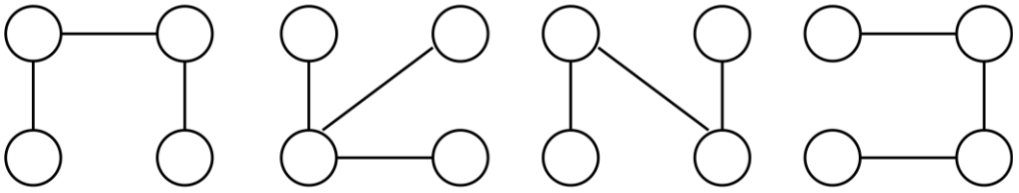
When CollapsingFind is uised the first Find(8) requires going up three links and then resetting two links. Total 13 movies requies for process all eight finds.

Spanning Tree:-

Let G=(V<E) be an undirected connected graph. A sub graph t=(V,E^1) of G is a spanning tree of G iff t is a tree.



A connected, undirected graph



Four of the spanning trees of the graph

Spanning Trees have many applications.
Example:-

It can be used to obtain an independent set of circuit equations for an electric network.
Any connected graph with n vertices must have at least n-1 edges and all connected graphs with n-1 edges are trees. If nodes of G represent cities and the edges represent possible communication links connecting two cities, then the minimum number of links needed to connect the n cities is n-1.
There are two basic algorithms for finding minimum-cost spanning trees, and both are greedy algorithms

→ Prim's Algorithm

→ Kruskal's Algorithm

Prim's Algorithm: Start with any *one node* in the spanning tree, and repeatedly add the cheapest edge, and the node it leads to, for which the node is not already in the spanning tree.

Kruskal's Algorithm: Start with *no* nodes or edges in the spanning tree, and repeatedly add the cheapest edge that does not create a cycle.

Connected Component:

Connected component of a graph can be obtained by using BFST (Breadth first search and traversal) and DFST (Dept first search and traversal). It is also called the spanning tree.

BFST (Breadth first search and traversal):

In BFS we start at a vertex V mark it as reached (visited).

The vertex V is at this time said to be unexplored (not yet discovered).

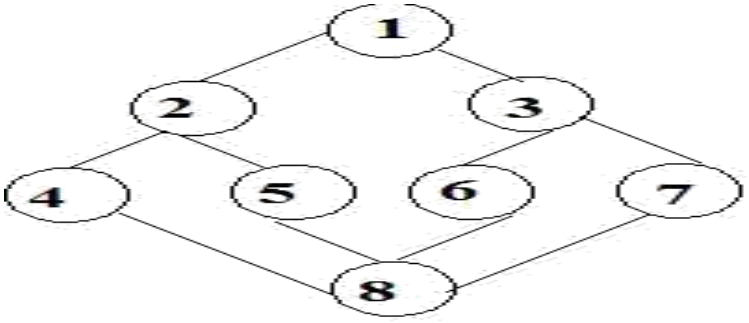
A vertex is said to been explored (discovered) by visiting all vertices adjacent from it.

All unvisited vertices adjacent from V are visited next.

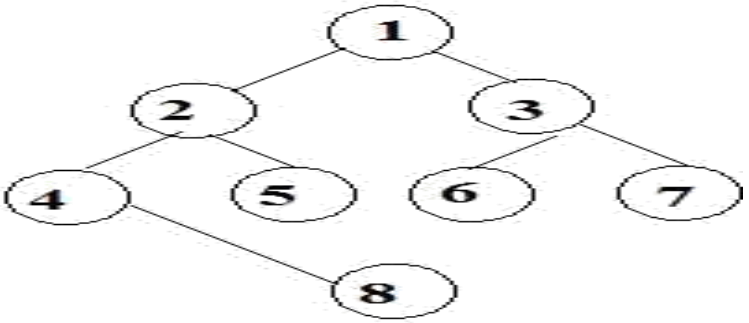
The first vertex on this list is the next to be explored.

Exploration continues until no unexplored vertex is left.

These operations can be performed by using Queue.



Undirected Graph G



BFS Spanning tree

This is also called connected graph or spanning tree.

Spanning trees obtained using BFS then it called breadth first spanning trees.

Algorithm for BFS to convert undirected graph G to Connected component or spanning tree.

Algorithm BFS(v)
// a bfs of G is begin at vertex v
// for any node I, visited[i]=1 if I has already been visited.
// the graph G, and array visited[] are global
{

```
U:=v; // q is a queue of unexplored vertices.
Visited[v]:=1;
Repeat{
For all vertices w adjacent from U do
If (visited[w]=0) then
{
Add w to q; // w is unexplored
Visited[w]:=1;
}
If q is empty then return; // No unexplored vertex.
Delete U from q; //Get 1st unexplored vertex.
} Until(false)
}
```

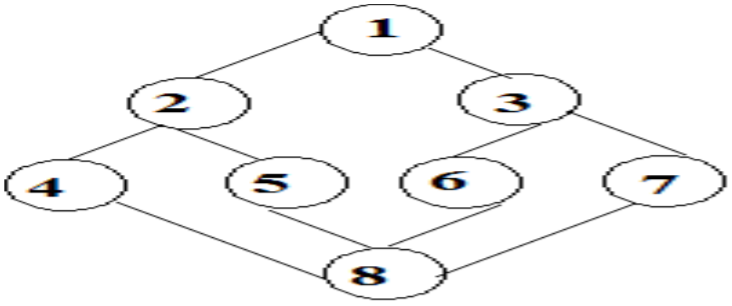
Maximum Time complexity and space complexity of G(n,e), nodes are in adjacency list. $T(n, e)=\theta(n+e)$
$S(n, e)=\theta(n)$

If nodes are in adjacency matrix then
$T(n, e)=\theta(n^2)$
$S(n, e)=\theta(n)$

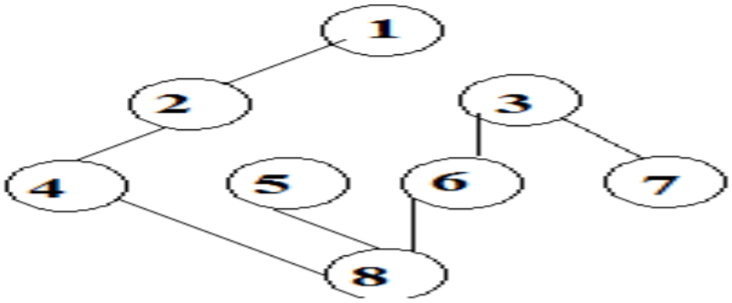
DFST(Dept first search and traversal).:

- Dfs different from bfs
- The exploration of a vertex v is suspended (stopped) as soon as a new vertex is reached.
  In this the exploration of the new vertex (example v) begins; this new vertex has been explored, the exploration of v continues.

Note: exploration start at the new vertex which is not visited in other vertex exploring and choose nearest path for exploring next or adjacent vertex.



Undirected Graph G



DFS(1) Spanning tree

Algorithm for DFS to convert undirected graph G to Connected component or spanning tree.

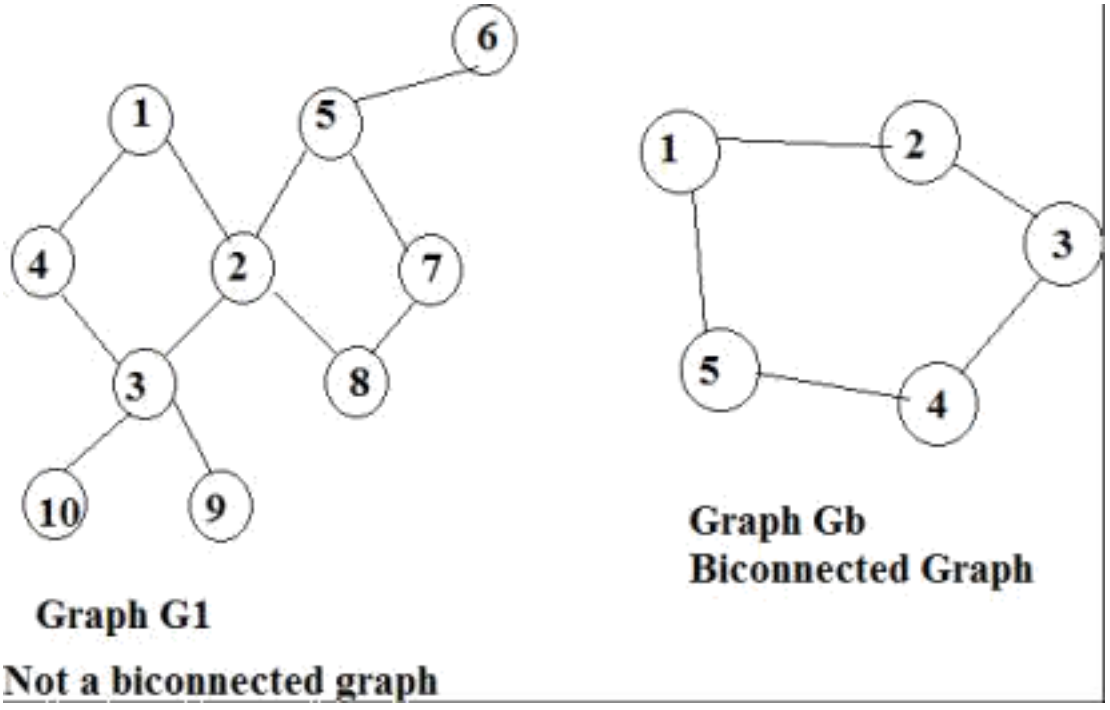
```
Algorithm dFS(v)
// a Dfs of G is begin at vertex v
// initially an array visited[] is set to zero.
//this algorithm visits all vertices reachable from v.
// the graph G, and array visited[] are global
{
Visited[v]:=1;
For each vertex w adjacent from v do
{
If (visited[w]=0) then DFS(w);
{
Add w to q; // w is
unexplored Visited[w]:=1;
}
}
```

Maximum Time complexity and space complexity of G(n,e), nodes are in adjacency list. $T(n, e)=\theta(n+e)$
$S(n, e)=\theta(n)$

If nodes are in adjacency matrix then
$T(n, e)=\theta(n^2)$
$S(n, e)=\theta(n)$

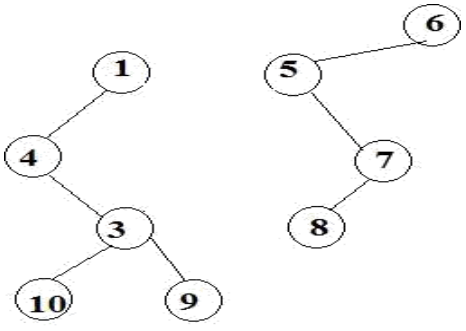
Bi-connected Components:

A graph G is biconnected, iff (if and only if) it contains no articulation point (joint or junction).
A vertex v in a connected graph G is an articulation point, if and only if (iff) the deletion of vertex v together with all edges incident to v disconnects the graph into two or more none empty components.



The presence of articulation points in a connected graph can be an undesirable(un wanted) feature in many cases.
For example
if G1 → Communication network with Vertex → communication stations. Edges Communication lines.

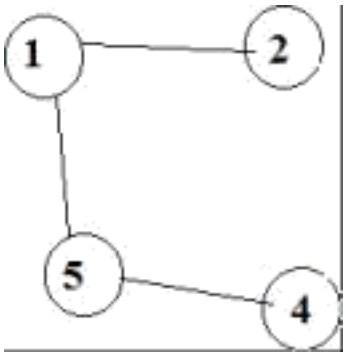
Then the failure of a communication station I that is an articulation point, then we loss the communication in between other stations. F
Form graph G1



After deleting vertex (2) (Here 2 is articulation point)

If the graph is bi-connected graph (means no articulation point) then if any station i fails, we can still communicate between every two stations not including station i.

From Graph Gb



There is an efficient algorithm to test whether a connected graph is biconnected. If the case of graphs that are not biconnected, this algorithm will identify all the articulation points.

Once it has been determined that a connected graph G is not biconnected, it may be desirable (suitable) to determine a set of edges whose inclusion makes the graph biconnected.

MODULE III:

Greedy method: General method, applications - Job sequencing with deadlines, 0/1 knapsack problem, Minimum cost spanning trees, Single source shortest path problem. **Dynamic Programming:** General method, applications-Matrix chain multiplication, Optimal binary search trees, 0/1 knapsack problem, All pairs shortest path problem, Travelling sales person problem, Reliability design.

Greedy Method:

The greedy method is perhaps (maybe or possible) the most straight forward design technique, used to determine a feasible solution that may or may not be optimal.

Feasible solution:- Most problems have n inputs and its solution contains a subset of inputs that satisfies a given constraint(condition). Any subset that satisfies the constraint is called feasible solution.

Optimal solution: To find a feasible solution that either maximizes or minimizes a given objective function. A feasible solution that does this is called optimal solution.

The greedy method suggests that an algorithm works in stages, considering one input at a time. At each stage, a decision is made regarding whether a particular input is in an optimal solution.

Greedy algorithms neither postpone nor revise the decisions (ie., no back tracking). **Example**: Kruskal's minimal spanning tree. Select an edge from a sorted list, check, decide, and never visit it again.

- **Application of Greedy Method:**
- Job sequencing with deadline
- 0/1 knapsack problem
- Minimum cost spanning trees

Single source shortest path problem.

Algorithm for Greedy method

Algorithm Greedy(a,n)
//a[1:n] contains the n inputs.
{
Solution :=0;
For i=1 to n do
{
X:=select(a);
If Feasible(solution, x) then
Solution :=Union(solution,x);
}
Return solution;
}

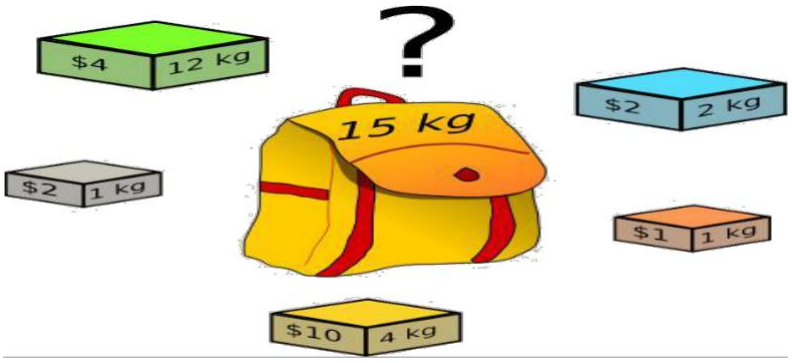
Selection → Function, that selects an input from a[] and removes it. The selected input's value is assigned to x.

Feasible → Boolean-valued function that determines whether x can be included into the solution vector.

Union → function that combines x with solution and updates the objective function.

Knapsack problem

The **knapsack problem** or **rucksack (bag) problem** is a problem in combinatorial optimization: Given a set of items, each with a mass and a value, determine the number of each item to include in a collection so that the total weight is less than or equal to a given limit and the total value is as large as possible



There are two versions of the problems

1. 0/1 knapsack problem
2. Fractional Knapsack problem

a. Bounded Knapsack problem.

b. Unbounded Knapsack problem.

Solutions to knapsack problems

- **Brute-force approach**:-Solve the problem with a straight farward algorithm
- **Greedy Algorithm**:- Keep taking most valuable items until maximum weight is reached or taking the largest value of eac item by calculating v_i=value$_i$/Size$_i$
- **Dynamic Programming**:- Solve each sub problem once and store their solutions in an array.

0/1 knapsack problem:

Let there be items, to where has a value and weight . The maximum weight that we can carry in the bag is *W*. It is common to assume that all values and weights are nonnegative. To simplify the representation, we also assume that the items are listed in increasing order of weight.

Maximize $\sum_{i=1}^{n} v_i x_i$ subject to $\sum_{i=1}^{n} w_i x_i \leqslant W, \qquad x_i \in \{0,1\}$

Maximize the sum of the values of the items in the knapsack so that the sum of the weights must be less than the knapsack's capacity.

Greedy algorithm for knapsack

Algorithm GreedyKnapsack(m,n)
// p[i:n] and [1:n] contain the profits and weights respectively
// if the n-objects ordered such that p[i]/w[i]>=p[i+1]/w[i+1], m → size of knapsack and x[1:n] → the solution vector
{
For i:=1 to n do x[i]:=0.0
U:=m;
For i:=1 to n do
{
if(w[i]>U) then
break; x[i]:=1.0;
U:=U-w[i];
}
If(i<=n) then x[i]:=U/w[i];
}

Ex: - Consider 3 objects whose profits and weights are defined as (P_1, P_2, P_3) = (25,24,15)

W_1,→W_2, W_3) = (18, 15, 10)

n=3 number of objects

m=20 → Bag capacity

Consider a knapsack of capacity 20. Determine the optimum strategy for placing the objects in to the knapsack. The problem can be solved by the greedy approach where in the inputs are arranged according to selection process (greedy strategy) and solve the problem in stages. The various greedy strategies for the problem could be as follows.

(x_1, x_2, x_3)	**$\sum x_i w_i$**	**$\sum x_i p_i$**
(1, 2/15, 0)	18x1+ 2/15 x15 = 20	25x1+ 2/15 x 24 = 28.2
(0, 2/3, 1)	15x2/3+10x1= 20	24x2/3 +15x1 = 31

(0,1,½)	1x15+ $\frac{1}{2}$x10 = 20	1x24+ $\frac{1}{2}$x15 = 31.5
(½,⅓,¼)	½ x 18+⅓ x15+ ¼ x10 = 16. 5	½ x 25+⅓ x24+ ¼ x15 = 12.5+8+3.75 = 24.25

Analysis: - If we do not consider the time considered for sorting the inputs then all of the three greedy strategies complexity will be O(n).

Job Sequence with Deadline:

There is set of n-jobs. For any job i, is a integer deadling di≥0 and profit Pi>0, the profit Pi is earned iff the job completed by its deadline.

To complete a job one had to process the job on a machine for one unit of time. Only one machine is available for processing jobs.

A feasible solution for this problem is a subset J of jobs such that each job in this subset can be completed by its deadline.

The value of a feasible solution J is the sum of the profits of the jobs in J, i.e.,

$\sum_{i \in j} P_i$ An optimal solution is a feasible solution with maximum value.

The problem involves identification of a subset of jobs which can be completed by its deadline. Therefore the problem suites the subset methodology and can be solved by the greedy method.

Ex: - Obtain the optimal sequence for the following jobs.

j_1 j_2 j_3 j_4

(P_1, P_2, P_3, P_4) = (100, 10, 15, 27)

(d_1, d_2, d_3, d_4) = (2, 1, 2, 1)

n =4

Feasible solution	**rocessing sequence**	**Value**
j_1 j_2 (1, 2)	(2,1)	100+10=110
(1,3)	(1,3) or (3,1)	100+15=115
(1,4)	(4,1)	100+27=127
(2,3)	(2,3)	10+15=25
(3,4)	(4,3)	15+27=42
(1)	(1)	100
(2)	(2)	10
(3)	(3)	15
(4)	(4)	27

In the example solution '3' is the optimal. In this solution only jobs 1&4 are processed and the value is 127. These jobs must be processed in the order j4 followed by j1. the process of job 4 begins at time 0 and ends at time 1. And the processing of job 1 begins at time 1 and ends at time2. Therefore both the jobs are completed within their deadlines. The optimization measure for determining the next job to be selected in to the solution is according to the profit. The next job to include is that which increases $\sum$pi the most, subject to the constraint that the resulting "j" is the feasible solution. Therefore the greedy strategy is to consider the jobs in decreasing order of profits.

The greedy algorithm is used to obtain an optimal solution.

We must formulate an optimization measure to determine how the next job is chosen.

```
algorithm js(d, j, n)
//d → dead line, j → subset of jobs ,n → total number of jobs
// d[i]≥1 1 ≤ i ≤ n are the dead lines,
// the jobs are ordered such that p[1]≥p[2]≥---≥p[n]
                                        → subset range//j[i]istheithjobintheoptimalsolution1≤i≤k,k
{
d[0]=j[0]=0;
j[1]=1;
k=1;
for i=2 to n do{
r=k;
while((d[j[r]]>d[i]) and [d[j[r]]≠r)) do
r=r-1;
if((d[j[r]]≤d[i]) and (d[i]> r)) then
{
for q:=k to (r+1) setp-1 do j[q+1]= j[q];
j[r+1]=i;
k=k+1;
}
}
return k;
}
```

Note: The size of sub set j must be less than equal to maximum deadline in given list.

Single Source Shortest Paths:

- Graphs can be used to represent the highway structure of a state or country with vertices representing cities and edges representing sections of highway.
- The edges have assigned weights which may be either the distance between the 2 cities connected by the edge or the average time to drive along that section of highway.
- For example if A motorist wishing to drive from city A to B then we must answer the following questions
  - Is there a path from A to B
  - If there is more than one path from A to B which is the shortest path
- The length of a path is defined to be the sum of the weights of the edges on that path. Given a directed graph G(V,E) with weight edge w(u,v). e have to find a shortest path from source vertex S∈v to every other vertex v1∈ V−S.

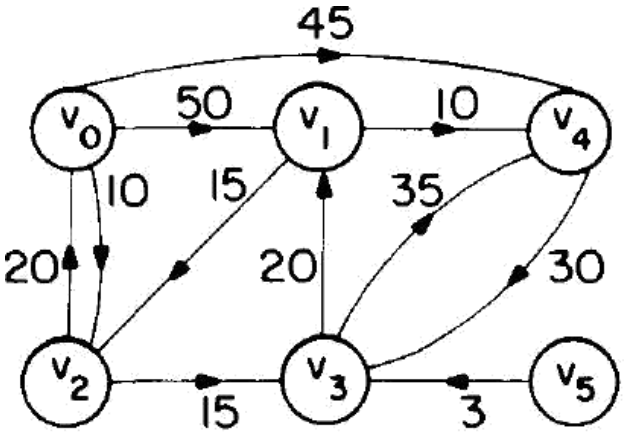
To find SSSP for directed graphs G(V,E) there are two different algorithms.

Bellman-Ford Algorithm

Dijkstra's algorithm

Bellman-Ford Algorithm:- allow –ve weight edges in input graph. This algorithm either finds a shortest path form source vertex **S∈V** to other vertex **v∈V** or detect a – ve weight cycles in G, hence no solution. If there is no negative weight cycles are reachable form source vertex **S∈V** to every other vertex **v∈V**

Dijkstra's algorithm:- allows only +ve weight edges in the input graph and finds a shortest path from source vertex **S∈V** to every other vertex **v∈V**.



	Path	*Length*
1)	$v_0\, v_2$	10
2)	$v_0\, v_2\, v_3$	25
3)	$v_0\, v_2\, v_3\, v_1$	45
4)	$v_0\, v_4$	45

Graph and shortest paths from v_0 to all destinations

Consider the above directed graph, if node 1 is the source vertex, then shortest path from 1 to 2 is 1,4,5,2. The length is 10+15+20=45.

To formulate a greedy based algorithm to generate the shortest paths, we must conceive of a multistage solution to the problem and also of an optimization measure.

This is possible by building the shortest paths one by one.

As an optimization measure we can use the sum of the lengths of all paths so far generated.

If we have already constructed 'i' shortest paths, then using this optimization measure, the next path to be constructed should be the next shortest minimum length path.

The greedy way to generate the shortest paths from Vo to the remaining vertices is to generate these paths in non-decreasing order of path length.

For this 1^{st}, a shortest path of the nearest vertex is generated. Then a shortest path to the 2^{nd} nearest vertex is generated and so on.

Algorithm for finding Shortest Path

```
Algorithm ShortestPath(v, cost, dist, n)
//dist[j], 1≤j≤n, is set to the length of the shortest path from vertex v to vertex j in graph
g with n-vertices.
// dist[v] is zero
{
for i=1 to n do{
s[i]=false;
dist[i]=cost[v,i];
}
s[v]=true;
dist[v]:=0.0; // put v in s
for num=2 to n do{
// determine n-1 paths from v
choose u form among those vertices not in s such that dist[u] is minimum.
s[u]=true; // put u in s
for (each w adjacent to u with s[w]=false) do
if(dist[w]>(dist[u]+cost[u, w])) then
dist[w]=dist[u]+cost[u, w];
}
}
```

Minimum Cost Spanning Tree:

SPANNING TREE: - A Sub graph 'n' of o graph 'G' is called as a spanning tree if

(i) It includes all the vertices of 'G'

(ii) It is a tree

Minimum cost spanning tree: For a given graph 'G' there can be more than one spanning tree. If weights are assigned to the edges of 'G' then the spanning tree which has the minimum cost of edges is called as minimal spanning tree.

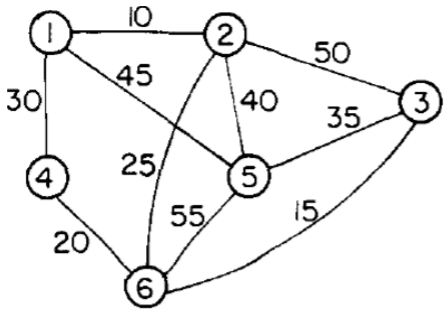
The greedy method suggests that a minimum cost spanning tree can be obtained by contacting the tree edge by edge. The next edge to be included in the tree is the edge that results in a minimum increase in the some of the costs of the edges included so far.

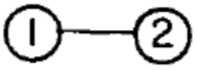
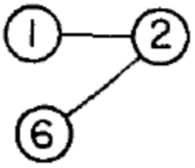
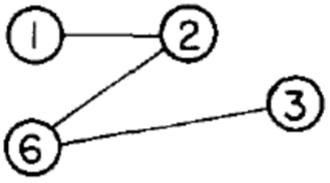
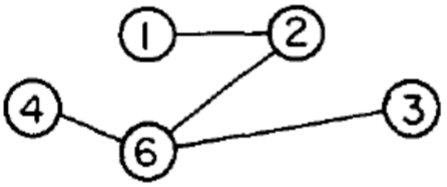
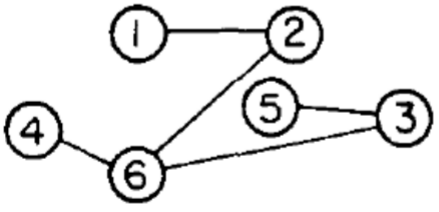
There are two basic algorithms for finding minimum-cost spanning trees, and both are greedy algorithms

→Prim's Algorithm

→Kruskal's Algorithm

Prim's Algorithm: Start with any *one node* in the spanning tree, and repeatedly add the cheapest edge, and the node it leads to, for which the node is not already in the spanning tree.



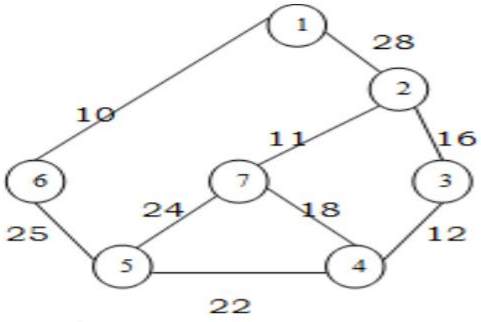
Edge	Cost	Spanning tree
(1,2)	10	
(2,6)	25	
(3,6)	15	
(6,4)	20	
(1,4)	reject	
(3,5)	35	

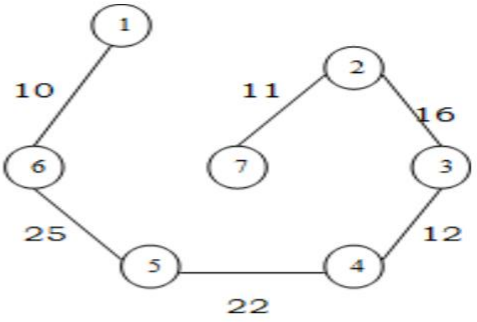
Stages in Prim's Algorithm

PRIM'S ALGORITHM: -

i) Select an edge with minimum cost and include in to the spanning tree.
ii) Among all the edges which are adjacent with the selected edge, select the one with minimum cost.
iii) Repeat step 2 until 'n' vertices and (n-1) edges are been included. And the sub graph obtained does not contain any cycles.

Notes: - At every state a decision is made about an edge of minimum cost to be included into the spanning tree. From the edges which are adjacent to the last edge included in the spanning tree i.e. at every stage the sub-graph obtained is a tree.



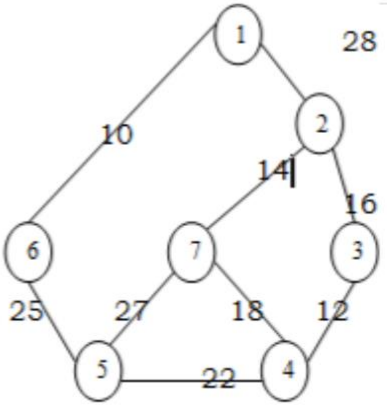


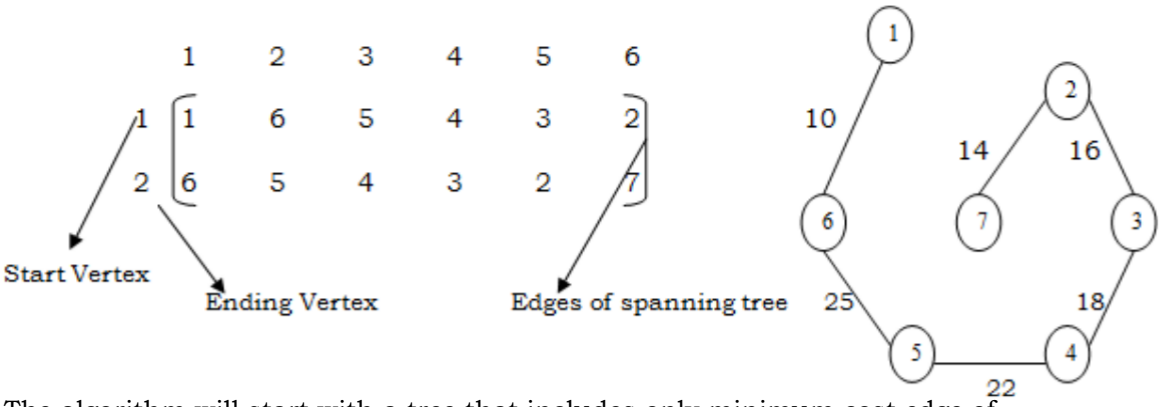
Prim's minimum spanning tree algorithm

```
Algorithm Prim (E, cost, n,t)
// E is the set of edges in G. Cost (1:n, 1:n) is the
// Cost adjacency matrix of an n vertex graph such that
// Cost (i,j) is either a positive real no. or ∞ if no edge (i,j) exists.
//A minimum spanning tree is computed and
//Stored in the array T(1:n-1, 2).
//(t (i, 1), + t(i,2)) is an edge in the minimum cost spanning tree. The final cost is
returned {
Let (k, l) be an edge with min cost in E Min
cost: = Cost (x,l);
T(1,1):= k; + (1,2):= l;
for i:= 1 to n do//initialize near
if (cost (i,l)<cost (i,k) then n east (i): l; else near
(i): = k;
near (k): = near (l): = 0; for i:
= 2 to n-1 do
{//find n-2 additional edges for t
let j be an index such that near (i) 0 & cost (j, near (i)) is
minimum; t (i,1): = j + (i,2): = near (j); min cost: = Min
cost + cost (j, near (j));
near (j): = 0;
for k:=1 to n do // update near ()
if ((near (k)   0) and (cost {k, near (k)) > cost
(k,j))) then near Z(k): = ji
}
return mincost;
}
```

The algorithm takes four arguments E: set of edges, cost is nxn adjacency matrix cost of (i,j)= +ve integer, if an edge exists between i&j otherwise infinity. 'n' is no/: of vertices. 't' is a (n-1):2matrix which consists of the edges of spanning tree.
E = { (1,2), (1,6), (2,3), (3,4), (4,5), (4,7), (5,6), (5,7), (2,7)
} n = {1,2,3,4,5,6,7)

Cost	1	2	3	4	5	6	7
1	α	28	α	α	α	10	α
2	28	α	16	α	α	α	14
3	α	10	α	12	α	α	α
4	α	α	12	α	22	α	18
5	α	α	α	22	α	25	24
6	10	α	α	α	25	α	α
7	α	14	α	18	24	α	α





i) The algorithm will start with a tree that includes only minimum cost edge of G. Then edges are added to this tree one by one.

ii) The next edge (i,j) to be added is such that i is a vertex which is already included in the treed and j is a vertex not yet included in the tree and cost of i,j is minimum among all edges adjacent to 'i'.

iii) With each vertex 'j' next yet included in the tree, we assign a value near 'j'. The value near 'j' represents a vertex in the tree such that cost (j, near (j)) is minimum among all choices for near (j)

iv) We define near (j):= 0 for all the vertices 'j' that are already in the tree.

v) The next edge to include is defined by the vertex 'j' such that (near (j)) 0 and cost of (j, near (j)) is minimum.

Analysis: -

The time required by the prince algorithm is directly proportional to the no/: of vertices. If a graph 'G' has 'n' vertices then the time required by prim's algorithm is **$O(n_2)$**

Kruskal's Algorithm: Start with *no* nodes or edges in the spanning tree, and repeatedly add the cheapest edge that does not create a cycle.

In Kruskals algorithm for determining the spanning tree we arrange the edges in the increasing order of cost.

i) All the edges are considered one by one in that order and deleted from the graph and are included in to the spanning tree.

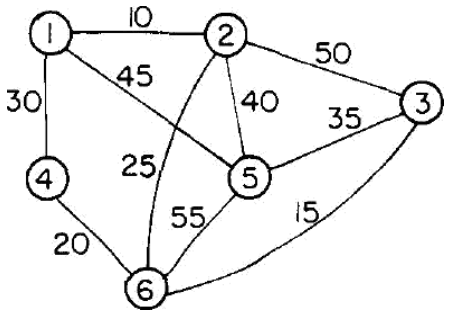
ii) At every stage an edge is included; the sub-graph at a stage need not be a tree. Infect it is a forest.

iii) At the end if we include 'n' vertices and n-1 edges without forming cycles then we get a single connected component without any cycles i.e. a tree with minimum cost.

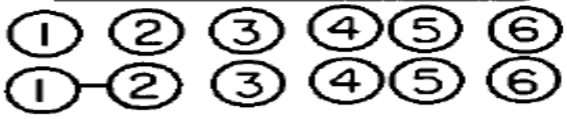
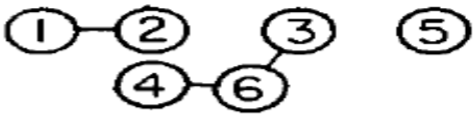
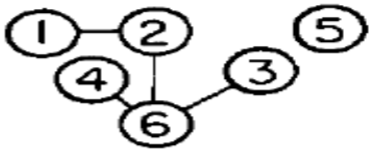
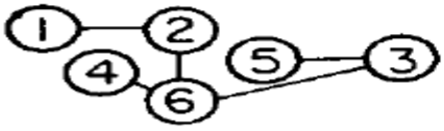
At every stage, as we include an edge in to the spanning tree, we get disconnected trees represented by various sets. While including an edge in to the spanning tree we need to check it does not form cycle. Inclusion of an edge (i,j) will form a cycle if i,j both are in same set. Otherwise the edge can be included into the spanning tree.

Kruskal minimum spanning tree algorithm

```
Algorithm Kruskal (E, cost, n,t)
//E is the set of edges in G. 'G' has 'n' vertices
//Cost {u,v} is the cost of edge (u,v) t is the set
//of edges in the minimum cost spanning tree
//The final cost is returned
{ construct a heap out of the edge costs using heapify;
for i:= 1 to n do parent (i):= -1 // place in different sets
 ertex is in different set                    {1} {1} {3}
i: = 0; min cost: = 0.0;
While (i<n-1) and (heap not empty))do
{
Delete a minimum cost edge (u,v) from the heaps; and reheapify using adjust; j:=
find (u); k:=find (v);
if (j  k) then
{ i: = 1+1;
+ (i,1)=u; + (i, 2)=v; mincost:
= mincost+cost(u,v); Union
(j,k);
}
}
(i n-1) then write ("No spanning tree");
else return mincost;
}
```



Consider the above graph of , Using Kruskal's method the edges of this graph are considered for inclusion in the minimum cost spanning tree in the order (1, 2), (3, 6), (4, 6), (2, 6), (1, 4), (3, 5), (2, 5), (1, 5), (2, 3), and (5, 6). This corresponds to the cost sequence 10, 15, 20, 25, 30, 35, 40, 45, 50, 55. The first four edges are included in T. The next edge to be considered is (I, 4). This edge connects two vertices already connected in T and so it is rejected. Next, the edge (3, 5) is selected and that completes the spanning tree.

Edge	Cast	Spanning Forest
(1,2)	10	
(3,6)	15	
(4,6)	20	
(2,6)	25	
(1,4)	30	(reject)
(3,5)	35	

Stages in Kruskal's algorithm

Analysis: - If the no/: of edges in the graph is given by /E/ then the time for Kruskals algorithm is given by 0 (|E| log |E|).

Dynamic Programming

Dynamic programming is a name, coined by Richard Bellman in 1955. Dynamic programming, as greedy method, is a powerful algorithm design technique that can be used when the solution to the problem may be viewed as the result of a sequence of decisions. In the greedy method we make irrevocable decisions one at a time, using a greedy criterion. However, in dynamic programming we examine the decision sequence to see whether an optimal decision sequence contains optimal decision subsequence.

When optimal decision sequences contain optimal decision subsequences, we can establish recurrence equations, called *dynamic-programming recurrence equations,* that enable us to solve the problem in an efficient way.

Dynamic programming is based on the principle of optimality (also coined by Bellman). The principle of optimality states that no matter whatever the initial state and initial decision are, the remaining decision sequence must constitute an optimal decision sequence with regard to the state resulting from the first decision. The principle implies that an optimal decision sequence is comprised of optimal decision subsequences. Since the principle of optimality may not hold for some formulations of some problems, it is necessary to verify that it does hold for the problem being solved. Dynamic programming cannot be applied when this principle does not hold.

The steps in a dynamic programming solution are:

- Verify that the principle of optimality holds
- Set up the dynamic-programming recurrence equations
- Solve the dynamic-programming recurrence equations for the value of the optimal solution.
- Perform a trace back step in which the solution itself is constructed.

5.1 MULTI STAGE GRAPHS

A multistage graph G = (V, E) is a directed graph in which the vertices are partitioned into $k \geq 2$ disjoint sets Vi, $1 \leq i \leq k$. In addition, if <u, v> is an edge in E, then u E Vi and v E Vi+1 for some i, $1 \leq i < k$.

Let the vertex ‘s’ is the source, and ‘t’ the sink. Let c (i, j) be the cost of edge <i, j>. The cost of a path from ‘s’ to ‘t’ is the sum of the costs of the edges on the path. The multistage graph problem is to find a minimum cost path from ‘s’ to ‘t’. Each set Vi defines a stage in the graph. Because of the constraints on E, every path from ‘s’ to ‘t’ starts in stage 1, goes to stage 2, then to stage 3, then to stage 4, and so on, and eventually terminates in stage k.

A dynamic programming formulation for a k-stage graph problem is obtained by first noticing that every s to t path is the result of a sequence of k – 2 decisions. The ith

decision involves determining which vertex in v_{i+1}, $1 \leq i \leq k - 2$, is to be on the path. Let c (i, j) be the cost of the path from source to destination. Then using the forward approach, we obtain:

cost (i, j) = min {c (j, l) + cost (i + 1, l)}
l c Vi + 1
<j, l> c E

ALGORITHM:

Algorithm Fgraph (G, k, n, p)
// The input is a k-stage graph G = (V, E) with n vertices // indexed in order or stages. E is a set of edges and c [i, j] // is the cost of (i, j). p [1 : k] is a minimum cost path.
{
cost [n] := 0.0;
for j:= n - 1 to 1 step – 1 do
{ // compute cost [j]
let r be a vertex such that (j, r) is an edge of G and c [j, r] + cost [r] is minimum; cost [j] := c [j, r] + cost [r];
d [j] := r:
}
p [1] := 1; p [k] := n; // Find a minimum cost path.
for j := 2 to k - 1 do p [j] := d [p [j - 1]];}

The multistage graph problem can also be solved using the backward approach. Let bp(i, j) be a minimum cost path from vertex s to j vertex in Vi. Let Bcost(i, j) be the cost of bp(i, j). From the backward approach we obtain:

st (i, j) = min { Bcost (i –1, l) + c (l, j)} l e Vi - 1
<l, j> e E

Algorithm Bgraph (G, k, n, p)
// Same function as Fgraph {
Bcost [1] := 0.0; for j := 2 to n do { / / C o m p u t e B c o s t [j] .
Let r be such that (r, j) is an edge of G and Bcost [r] + c [r, j] is minimum; Bcost [j] := Bcost [r] + c [r, j]; D [j] := r;

} //find a minimum cost path p [1] := 1; p [k] := n;

for j:= k - 1 to 2 do p [j] := d [p [j + 1]];
}

Complexity Analysis:

The complexity analysis of the algorithm is fairly straightforward. Here, if G has ~E~ edges, then the time for the first for loop is CJ (V~ +~E).

EXAMPLE 1:

Find the minimum cost path from s to t in the multistage graph of five stages shown below. Do this first using forward approach and then using backward approach.

FORWARD APPROACH:

We use the following equation to find the minimum cost path from s to t: cost (i,
min {c (j, l) + cost (i + 1, l)} l c Vi + 1
<j, l> c E
cost (1, 1) = min {c (1, 2) + cost (2, 2), c (1, 3) + cost (2, 3), c (1, 4) + cost (2, 4), c (1, 5) + cost (2, 5)}
= min {9 + cost (2, 2), 7 + cost (2, 3), 3 + cost (2, 4), 2 + cost (2, 5)}

Now first starting with,

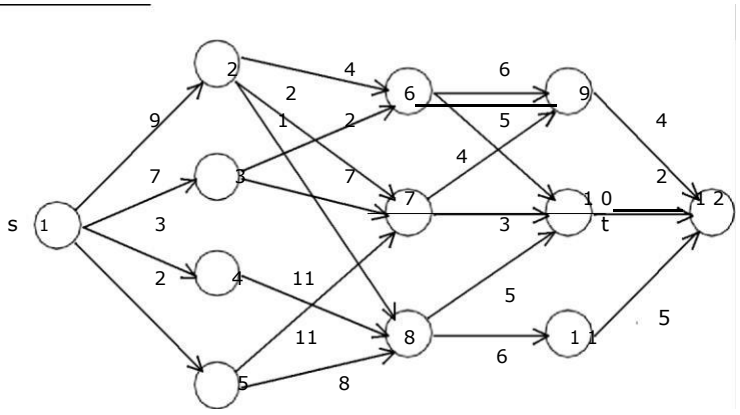
cost (2, 2) = min{c (2, 6) + cost (3, 6), c (2, 7) + cost (3, 7), c (2, 8) + cost (3, 8)} = min {4 + cost (3, 6), 2 + cost (3, 7), 1 + cost (3, 8)}

cost (3, 6) = min {c (6, 9) + cost (4, 9), c (6, 10) + cost (4, 10)}
= min {6 + cost (4, 9), 5 + cost (4, 10)}

cost (4, 9) = min {c (9, 12) + cost (5, 12)} = min {4 + 0) = 4 cost (4,

10) = min {c (10, 12) + cost (5, 12)} = 2

Therefore, cost (3, 6) = min {6 + 4, 5 + 2} = 7

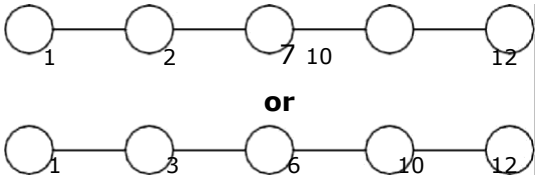


cost (3, 7) = min {c (7, 9) + cost (4, 9) , c (7, 10) + cost (4, 10)}
= min {4 + cost (4, 9), 3 + cost (4, 10)}

cost (4, 9) = min {c (9, 12) + cost (5, 12)} = min {4 + 0} = 4 Cost

(4, 10) =

The path is min



{c (10,

2) + cost (5, 12)} = min {2 + 0} = 2 Therefore, cost (3, 7) = min {4 + 4, 3

+ 2} = min {8, 5} = 5

cost (3, 8) = min {c (8, 10) + cost (4, 10), c (8, 11) + cost (4, 11)}
= min {5 + cost (4, 10), 6 + cost (4 + 11)}

cost (4, 11) = min {c (11, 12) + cost (5, 12)} = 5

Therefore, cost (3, 8) = min {5 + 2, 6 + 5} = min {7, 11} = 7

Therefore, cost (2, 2) = min {4 + 7, 2 + 5, 1 + 7} = min {11, 7, 8} = 7

Therefore, cost (2, 3) = min {c (3, 6) + cost (3, 6), c (3, 7) + cost (3, 7)}
= min {2 + cost (3, 6), 7 + cost (3, 7)}
= min {2 + 7, 7 + 5} = min {9, 12} = 9

cost (2, 4) = min {c (4, 8) + cost (3, 8)} = min {11 + 7} = 18 cost (2, 5)
= min {c (5, 7) + cost (3, 7), c (5, 8) + cost (3, 8)} = min {11 + 5, 8 +
7} = min {16, 15} = 15

Therefore, cost (1, 1) = min {9 + 7, 7 + 9, 3 + 18, 2 + 15} =
min {16, 16, 21, 17} = 16

The minimum cost path is 16.

BACKWARD APPROACH:

We use the following equation to find the minimum cost path from t to s: Bcost (i, J) = min

{Bcost (i – 1, l) + c (l, J)}

l c vi – 1
<l, j> c E

Bcost (5, 12) = min {Bcost (4, 9) + c (9, 12), Bcost (4, 10) + c (10, 12),
Bcost (4, 11) + c (11, 12)}
= min {Bcost (4, 9) + 4, Bcost (4, 10) + 2, Bcost (4, 11) + 5}

Bcost (4, 9) = min {Bcost (3, 6) + c (6, 9), Bcost (3, 7) + c (7,
9)} = min {Bcost (3, 6) + 6, Bcost (3, 7) + 4}

Bcost (3, 6) = min {Bcost (2, 2) + c (2, 6), Bcost (2, 3) + c (3,
6)} = min {Bcost (2, 2) + 4, Bcost (2, 3) + 2}

Bcost (2, 2) = min {Bcost (1, 1) + c (1, 2)} = min {0 + 9} = 9 Bcost (2, 3) = min

{Bcost (1, 1) + c (1, 3)} = min {0 + 7} = 7 Bcost (3, 6) = min {9 + 4, 7 + 2} =

min {13, 9} = 9

Bcost (3, 7) = min {Bcost (2, 2) + c (2, 7), Bcost (2, 3) + c (3, 7), Bcost (2, 5) + c (5, 7)}

Bcost (2, 5) = min {Bcost (1, 1) + c (1, 5)} = 2

Bcost (3, 7) = min {9 + 2, 7 + 7, 2 + 11} = min {11, 14, 13} = 11 Bcost (4, 9) = min {9

+ 6, 11 + 4} = min {15, 15} = 15

Bcost (4, 10) = min {Bcost (3, 6) + c (6, 10), Bcost (3, 7) + c (7, 10),
Bcost (3, 8) + c (8, 10)}

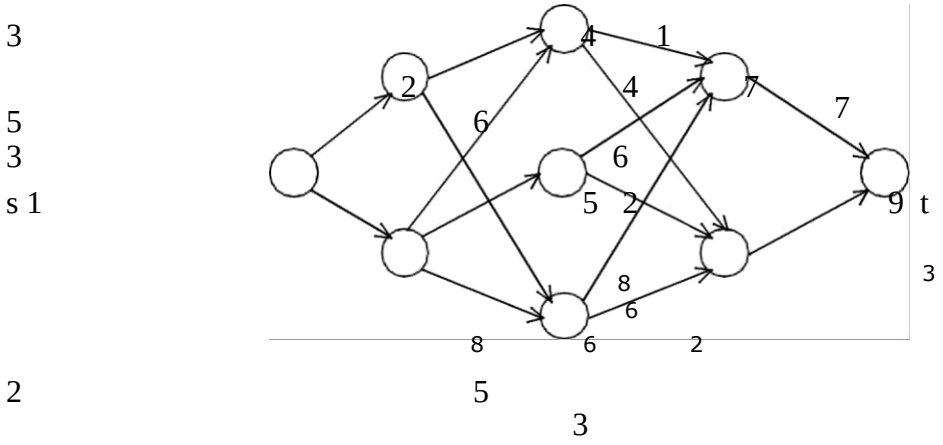
Bcost (3, 8) = min {Bcost (2, 2) + c (2, 8), Bcost (2, 4) + c (4,
8), Bcost (2, 5) + c (5, 8)}

Bcost (2, 4) = min {Bcost (1, 1) + c (1, 4)} = 3
Bcost (3, 8) = min {9 + 1, 3 + 11, 2 + 8} = min {10, 14, 10} = 10 Bcost (4, 10) = min {9 +
5, 11 + 3, 10 + 5} = min {14, 14, 15) = 14
Bcost (4, 11) = min {Bcost (3, 8) + c (8, 11)} = min {Bcost (3, 8) + 6} = min {10 + 6} = 16

Bcost (5, 12) = min {15 + 4, 14 + 2, 16 + 5} = min {19, 16, 21} = 16. **EXAMPLE**

2:

Find the minimum cost path from s to t in the multistage graph of five stages shown below. Do this first using forward approach and then using backward approach.



SOLUTION: FORWARD

APPROACH:

cost (i, J) = min {c (j, l) + cost (i + 1, l)}
l c Vi + 1
<J, l> EE

cost (1, 1) = min {c (1, 2) + cost (2, 2), c (1, 3) + cost (2, 3)}
= min {5 + cost (2, 2), 2 + cost (2, 3)}

cost (2, 2) = min {c (2, 4) + cost (3, 4), c (2, 6) + cost (3, 6)} =
min {3+ cost (3, 4), 3 + cost (3, 6)}

cost (3, 4) = min {c (4, 7) + cost (4, 7), c (4, 8) + cost (4, 8)}
= min {(1 + cost (4, 7), 4 + cost (4, 8)}

cost (4, 7) = min {c (7, 9) + cost (5, 9)} = min {7 + 0) = 7 cost (4, 8)

= min {c (8, 9) + cost (5, 9)} = 3

Therefore, cost (3, 4) = min {8, 7} = 7

cost (3, 6) = min {c (6, 7) + cost (4, 7), c (6, 8) + cost (4, 8)}
= min {6 + cost (4, 7), 2 + cost (4, 8)} = min {6 + 7, 2 + 3} = 5

Therefore, cost (2, 2) = min {10, 8} = 8

cost (2, 3) = min {c (3, 4) + cost (3, 4), c (3, 5) + cost (3, 5), c (3, 6) + cost (3,6)}

cost (3, 5) = min {c (5, 7) + cost (4, 7), c (5, 8) + cost (4, 8)}= min {6 + 7, 2 + 3} = 5

Therefore, cost (2, 3) = min {13, 10, 13} = 10

cost (1, 1) = min {5 + 8, 2 + 10} = min {13, 12} = 12

BACKWARD APPROACH:

Bcost (i, J) = min {Bcost (i – 1, l) = c (l, J)}
l E vi – 1
<l ,j>E E

Bcost (5, 9) = min {Bcost (4, 7) + c (7, 9), Bcost (4, 8) + c (8, 9)} =
min {Bcost (4, 7) + 7, Bcost (4, 8) + 3}

Bcost (4, 7) = min {Bcost (3, 4) + c (4, 7), Bcost (3, 5) + c (5,
7), Bcost (3, 6) + c (6, 7)}
= min {Bcost (3, 4) + 1, Bcost (3, 5) + 6, Bcost (3, 6) + 6}

Bcost (3, 4) = min {Bcost (2, 2) + c (2, 4), Bcost (2, 3) + c (3,
4)} = min {Bcost (2, 2) + 3, Bcost (2, 3) + 6}

Bcost (2, 2) = min {Bcost (1, 1) + c (1, 2)} = min {0 + 5} = 5

Bcost (2, 3) = min (Bcost (1, 1) + c (1, 3)} = min {0 + 2} = 2

Therefore, Bcost (3, 4) = min {5 + 3, 2 + 6} = min {8, 8} = 8

Bcost (3, 5) = min {Bcost (2, 3) + c (3, 5)} = min {2 + 5} = 7

Bcost (3, 6) = min {Bcost (2, 2) + c (2, 6), Bcost (2, 3) + c (3, 6)} =
min {5+5,2+8}=10

Therefore, Bcost (4, 7) = min {8 + 1, 7 + 6, 10 + 6} = 9

Bcost (4, 8) = min {Bcost (3, 4) + c (4, 8), Bcost (3, 5) + c (5, 8), Bcost (3,
6) + c (6, 8)}
= min {8 + 4, 7 + 2, 10 + 2} = 9

Therefore, Bcost (5, 9) = min {9 + 7, 9 + 3} = 12 **All**

pairs shortest paths

In the all pairs shortest path problem, we are to find a shortest path between every pair of vertices in a directed graph G. That is, for every pair of vertices (i, j), we are to find a shortest path from i to j as well as one from j to i. These two paths are the same when G is undirected.

When no edge has a negative length, the all-pairs shortest path problem may be solved by using Dijkstra's greedy single source algorithm n times, once with each of the n vertices as the source vertex.

The all pairs shortest path problem is to determine a matrix A such that A (i, j) is the length of a shortest path from i to j. The matrix A can be obtained by solving n single-source

problems using the algorithm shortest Paths. Since each application of this procedure requires O (n^2) time, the matrix A can be obtained in O (n^3) time.

The dynamic programming solution, called Floyd's algorithm, runs in O (n^3) time. Floyd's algorithm works even when the graph has negative length edges (provided there are no negative length cycles).

The shortest i to j path in G, $i \neq j$ originates at vertex i and goes through some intermediate vertices (possibly none) and terminates at vertex j. If k is an intermediate vertex on this shortest path, then the subpaths from i to k and from k to j must be shortest paths from i to k and k to j, respectively. Otherwise, the i to j path is not of minimum length. So, the principle of optimality holds. Let A^k (i, j) represent the length of a shortest path from i to j going through no vertex of index greater than k, we obtain:

$Ak(i, j) = \{\min \{\min \{A^{k-1}(i, k) + A^{k-1}(k, j)\}, c(i, j)\}\ 1<k<n$

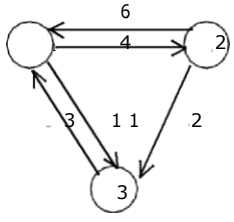
Algorithm All Paths (Cost, A, n)
// cost [1:n, 1:n] is the cost adjacency matrix of a graph which
// n vertices; A [I, j] is the cost of a shortest path from vertex
// i to vertex j. cost [i, i] = 0.0, for $1 \leq i \leq n$.
{
for i := 1 to n do
for j:= 1 to n do
A [i, j] := cost [i, j]; // copy cost into A. for k := 1 to n do

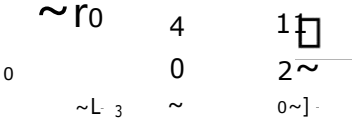
for i := 1 to n do
for j := 1 to n do
A [i, j] := min (A [i, j], A [i, k] + A [k, j]);
}

Complexity Analysis: A Dynamic programming algorithm based on this recurrence involves in calculating n+1 matrices, each of size n x n. Therefore, the algorithm has a complexity of O (n^3).

Example 1:

Given a weighted digraph G = (V, E) with weight. Determine the length of the shortest path between all pairs of vertices in G. Here we assume that there are no cycles with zero or negative cost.





General formula: min $\{A^{k-1}(i, k) + A^{k-1}(k, j)\}, c(i, j)\}$
$1<k<n$

Solve the problem for different values of k = 1, 2 and 3 **Step 1**:

Solving the equation for, k = 1;

A1 (1, 1) = min $\{(A^{o}(1, 1) + A^{o}(1, 1)), c(1, 1)\}$ = min {0 + 0, 0} = 0 A1 (1,
2) = min $\{(A^{o}(1, 1) + A^{o}(1, 2)), c(1, 2)\}$ = min {(0 + 4), 4} = 4
A1 (1, 3) = min $\{(A^{o}(1, 1) + A^{o}(1, 3)), c(1, 3)\}$ = min {(0 + 11), 11} = 11 A1 (2,
1) = min $\{(A^{o}(2, 1) + A^{o}(1, 1)), c(2, 1)\}$ = min {(6 + 0), 6} = 6
A1 (2, 2) = min $\{(A^{o}(2, 1) + A^{o}(1, 2)), c(2, 2)\}$ = min {(6 + 4), 0)} = 0 A1 (2,
3) = min $\{(A^{o}(2, 1) + A^{o}(1, 3)), c(2, 3)\}$ = min {(6 + 11), 2} = 2 A1 (3, 1)
= min $\{(A^{o}(3, 1) + A^{o}(1, 1)), c(3, 1)\}$ = min {(3 + 0), 3} = 3 A1 (3, 2) =
min $\{(A^{o}(3, 1) + A^{o}(1, 2)), c(3, 2)\}$ = min {(3 + 4), oc} = 7 A1 (3, 3) = min
$\{(A^{o}(3, 1) + A^{o}(1, 3)), c(3, 3)\}$ = min {(3 + 11), 0} = 0

$A_{(1)}$ =

~0	4	11
~6	0	2~
~L3	7	0~U

Step 2: Solving the equation for, K = 2;

A_2 (1, 1) = min $\{(A^{1}(1, 2) + A^{1}(2, 1), c(1, 1)\}$ = min {(4 + 6), 0} + A^{1} = 0
A_2 (1, 2) = min $\{(A^{1}(1, 2)$ (2, 2), c (1, 2)} = min {(4 + 0), 4} + A^{1} (2, = 4
A_2 (1, 3) = min $\{(A^{1}(1, 2)$ 3), c (1, 3)} = min {(4 + 2), 11} = 6
A_2 (2, 1) = min {(A (2, 2) + A (2, 1), c (2, 1)} = min {(0 + 6), 6} = 6
A_2 (2, 2) = min {(A (2, 2) + A (2, 2), c (2, 2)} = min {(0 + 0), 0} = 0
A_2 (2, 3) = min {(A (2, 2) + A (2, 3), c (2, 3)} = min {(0 + 2), 2} = 2
A_2 (3, 1) = min {(A (3, 2) + A (2, 1), c (3, 1)} = min {(7 + 6), 3} = 3
A_2 (3, 2) = min {(A (3, 2) + A (2, 2), c (3, 2)} = min {(7 + 0), 7} = 7
A_2 (3, 3) = min {(A (3, 2) + A (2, 3), c (3, 3)} = min {(7 + 2), 0} = 0

$A_{(2)}$ =

~0	4	6 1
~6	0	2~
~L3	7	0~~

Step 3: Solving the equation for, k = 3;

A3 (1, 1) = min {A^2 (1, 3) + A^2 (3, 1), c (1, 1)} = min {(6 + 3), 0} = 0

A3 (1, 2) = min {A^2 (1, 3) + A^2 (3, 2), c (1, 2)} = min {(6 + 7), 4} = 4

A3 (1, 3) = min {A^2 (1, 3) + A^2 (3, 3), c (1, 3)} = min {(6 + 0), 6} = 6

A3 (2, 1) = min {A^2 (2, 3) + A^2 (3, 1), c (2, 1)} = min {(2 + 3), 6} = 5

A3 (2, 2) = min {A^2 (2, 3) + A^2 (3, 2), c (2, 2)} = min {(2 + 7), 0} = 0

A3 (2, 3) = min {A^2 (2, 3) + A^2 (3, 3), c (2, 3)} = min {(2 + 0), 2} = 2

A3 (3, 1) = min {A^2 (3, 3) + A^2 (3, 1), c (3, 1)} = min {(0 + 3), 3} = 3

A3 (3, 2) = min {A^2 (3, 3) + A^2 (3, 2), c (3, 2)} = min {(0 + 7), 7} = 7

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A3 (3, 3) = min {A^2 (3, 3) + A^2 (3, 3), c (3, 3)} = min {(0 + 0), 0} = 0

$A_{(3)}$= ~0 4 6 ~
0 ~ ~
2
~~5~~3 7 0 ~]

TRAVELLING SALESPERSON PROBLEM

Let G = (V, E) be a directed graph with edge costs Cij. The variable cij is defined such that cij > 0 for all I and j and cij = a if < i, j> o E. Let |V| = n and assume n > 1. A tour of G is a directed simple cycle that includes every vertex in V. The cost of a tour is the sum of the cost of the edges on the tour. The traveling sales person problem is to find a tour of minimum cost. The tour is to be a simple path that starts and ends at vertex 1.

Let g (i, S) be the length of shortest path starting at vertex i, going through all vertices in S, and terminating at vertex 1. The function g (1, V – {1}) is the length of an optimal salesperson tour. From the principal of optimality it follows that:

g(1, V - {1 }) = 2 ~ *k* ~ *n* ~*c*1*k* ~ *g* ~ *k, V* ~ ~ 1, *k* ~~ -- 1

~

min

-- 2

Generalizing equation 1, we obtain (for i o S)

g (i, S) = min{*ci j*

j E*S*

The Equation can be solved for g (1, V – 1}) if we know g (k, V – {1, k}) for all choices of k.

Complexity Analysis:

For each value of |S| there sets S of +*g i,S- j* are n–1 choices for i. The number of distinct

~*n*-2~

size k not including 1 and i is I *k* ~ . ~ ~

~ ~

Hence, the total number of g (i, S)'s to be computed before computing g (1, V – {1}) is:

~*n*-2~

~ ~ *n* ~ 1 ~ ~

~ *k* ~

k ~ 0 ~ ~

To calculate this sum, we use the binominal theorem:

[((*n* - 2) ((*n* - 2) ((*n* - 2) ((*n* - 2)1

n - 1 (n–1)111 11+**i** iI+ ii iI+----~~~ ~

~~ 0) ~1) ~ 2) ~(*n*~ 2)~~

According to the binominal theorem:

[((*n* - 2) ((*n* - 2) ((*n* - 2 ((*n* - 2)1

il 11+**i** iI+**i** ~~~~~~~~~~ ~~~=2n-2

~~0~~1~~ 2 ~ ~(*n* - 2))]

Therefore,

n - 1 ~ *n* _ 2'

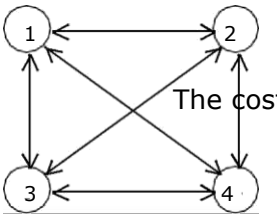
~ (*n* _ 1 ~ ~~*k* ~ = (*n* - 1) 2*n* ~ 2

This is Φ $(n\ 2^{n-2})$, so there are exponential number of calculate. Calculating one g (i, S) require finding the minimum of at most n quantities. Therefore, the entire algorithm is Φ $(n^2\ 2^{n-2})$. This is better than enumerating all n! different tours to find the best one. So, we have traded on exponential growth for a much smaller exponential growth.

The most serious drawback of this dynamic programming solution is the space needed, which is $O(n\,2^n)$. This is too large even for modest values of n.

Example 1:

For the following graph find minimum cost tour for the traveling salesperson problem:



The cost adjacency matrix =

$$\begin{bmatrix} 0 & 10 & 15 & 20 \\ 5 & 0 & 9 & 10 \\ 6 & 13 & 0 & 12 \\ 8 & 9 & 0 & 0 \end{bmatrix}$$

More generally writing:

$$g(i, s) = \min \{c_{ij} + g(J, s - \{J\})\} \qquad - \quad (2)$$

Clearly, $g(i, T) = c_{i1}$, $1 \le i \le n$. So,

$g(2, T) = C_{21} = 5$

$g(3, T) = C_{31} = 6$

$g(4, \sim) = C_{41} = 8$

Using equation – (2) we obtain:

$g(1, \{2, 3, 4\}) = \min \{c_{12} + g(2, \{3, 4\}), c_{13} + g(3, \{2, 4\}), c_{14} + g(4, \{2, 3\})\}$

$g(2, \{3, 4\}) = \min \{c_{23} + g(3, \{4\}), c_{24} + g(4, \{3\})\}$
$= \min \{9 + g(3, \{4\}), 10 + g(4, \{3\})\}$

$g(3, \{4\}) = \min \{c_{34} + g(4, T)\} = 12 + 8 = 20$

$g(4, \{3\}) = \min \{c_{43} + g(3, \sim)\} = 9 + 6 = 15$

Therefore, g (2, {3, 4}) = min {9 + 20, 10 + 15} = min {29, 25} = 25 g

(3, {2, 4}) = min {(c32 + g (2, {4}), (c34 + g (4, {2})}

g (2, {4}) = min {c24 + g (4, T)} = 10 + 8 = 18

g (4, {2}) = min {c42 + g (2, ~)} = 8 + 5 = 13

Therefore, g (3, {2, 4}) = min {13 + 18, 12 + 13} = min {41, 25} = 25

g (4, {2, 3}) = min {c42 + g (2, {3}), c43 + g (3, {2})} g
(2, {3}) = min {c23 + g (3, ~} = 9 + 6 = 15

g (3, {2}) = min {c32 + g (2, T} = 13+ 5 = 18

Therefore, g (4, {2, 3}) = min {8 + 15, 9 + 18} = min {23, 27} = 23

g (1, {2, 3, 4}) = min {c12 + g (2, {3, 4}), c13 + g (3, {2, 4}), c14 + g (4, {2, 3})} = min {10 + 25, 15 + 25, 20 + 23} = min {35, 40, 43} = 35

The optimal tour for the graph has length = 35 The optimal

tour is: 1, 2, 4, 3, 1.

OPTIMAL BINARY SEARCH TREE

Let us assume that the given set of identifiers is {a1, . . . , an} with a1 < a2 < < an.
Let p (i) be the probability with which we search for ai. Let q (i) be the probability that the identifier x being searched for is such that ai < x < ai+1, 0 ≤ i ≤ n (assume a0 = - ~ and an+1 = +oc). We have to arrange the identifiers in a binary search tree in a way that minimizes the expected total access time.
In a binary search tree, the number of comparisons needed to access an element at depth 'd' is d + 1, so if 'ai' is placed at depth 'di', then we want to minimize:

$$\sum_{i=1}^{n} Pi\,(1 + di)\ .$$

Let P (i) be the probability with which we shall be searching for 'ai'. Let Q (i) be the probability of an un-successful search. Every internal node represents a point where a successful search may terminate. Every external node represents a point where an unsuccessful search may terminate.

The expected cost contribution for the internal node for 'ai' is:

$P\,(i) * level\,(ai)$.

Unsuccessful search terminate with I = 0 (i.e at an external node). Hence the cost contribution for this node is:

Q (i) * level ((Ei) - 1)

The expected cost of binary search tree is:

$$\sim \sum_{i=1}^{n} P(i) * level(ai) + \sim \sum_{i=1}^{n} Q(i) * level((Ei) - 1)$$

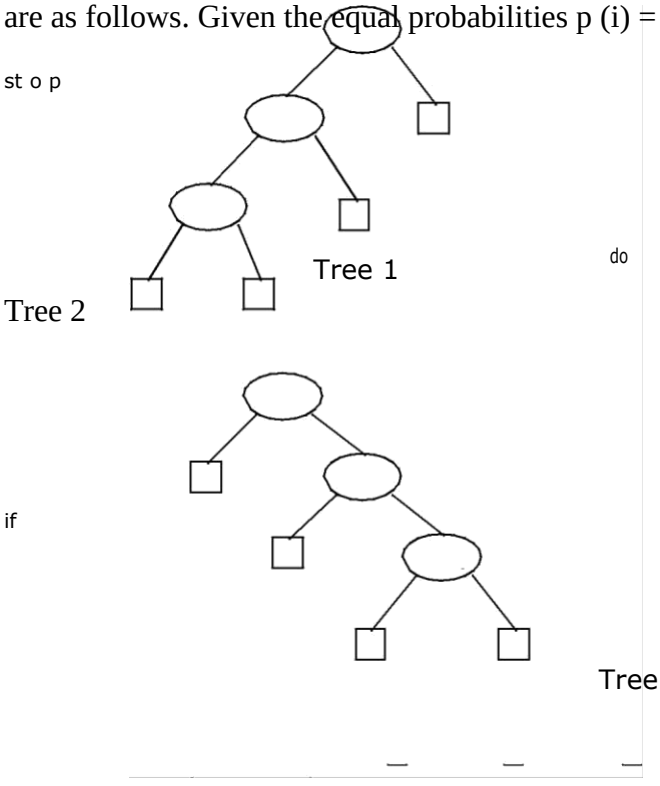
Given a fixed set of identifiers, we wish to create a binary search tree organization. We may expect different binary search trees for the same identifier set to have different performance characteristics.

The computation of each of these c(i, j)'s requires us to find the minimum of m quantities. Hence, each such c(i, j) can be computed in time O(m). The total time for all c(i, j)'s with j – i = m is therefore $O(nm - m^2)$.

The total time to evaluate all the c(i, j)'s and r(i, j)'s is therefore:

$\sim (nm - m^2) = O(n^3)$ $1 < m < n$

Example 1: The possible binary search trees for the identifier set (a1, a2, a3) = (do, if, stop) are as follows. Given the equal probabilities p (i) = Q (i) = 1/7 for all i, we have:



Cost (tree # 1) = $\sim \left(1 \times 1 + 1 \times 2 + 1 \times 3 \sim \right) + (\sim 1 \times 1) \sim 7$

$\sim 7 \quad 7 \quad 7$

1+2+31+2+3+36+9 15

Cost (tree # 3) = ~ 1 x 1 + 1 x 2 + 1 x 3~ ~ + (1 x 1 + 1 x 2 + 1 x 3 + 1 x 3~ ~
~ ~ 7 7 7) ~ 7 7 7 7)

=1+2+3 1+2+3+3 6+9 15
7 + 7 ~ (1 x ~1+1

Cost (tree # 4) = ~ 1 x 1 + 1 x 2 ~ 1 x 3~ ~ ~ ~ x 2 + 1 x 3 + 1 x 3~ ~
~ ~ 7 7) ~ 7 7 7 7)
~ 7

1+2+3+3 6 + 9 15
= 1+ 2 +3 7

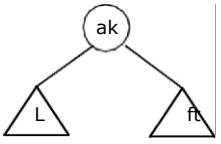
7

Cost (tree # 2) = (1 x 1 + 1 1 x 2~ (~1x 2 + 1 x 2 + 1 x 2 + 1 x2~
~ 7 7 x2 + 7) ~ 7 7 7 7)

=1+2+2 2+2+2+2 5+8 13
7 + 7 ~ 7 ~ 7

Huffman coding tree solved by a greedy algorithm has a limitation of having the data only at the leaves and it must not preserve the property that all nodes to the left of the root have keys, which are less etc. Construction of an optimal binary search tree is harder, because the data is not constrained to appear only at the leaves, and also because the tree must satisfy the binary search tree property and it must preserve the property that all nodes to the left of the root have keys, which are less.

A dynamic programming solution to the problem of obtaining an optimal binary search tree can be viewed by constructing a tree as a result of sequence of decisions by holding the principle of optimality. A possible approach to this is to make a decision as which of the ai's be arraigned to the root node at 'T'. If we choose 'ak' then is clear that the internal nodes for a1, a2, ak-1 as well as the external nodes for the classes Eo, E1,
. . . . . Ek-1 will lie in the left sub tree, L, of the root. The remaining nodes will be in the right subtree, ft. The structure of an optimal binary search tree is:



$$\text{Cost (L)} = \sum_{i=1}^{k} P(i) * level(a_i) + \sum_{i=0}^{k} Q(i) * level(E_i) - 1$$

$$\text{Cost (ft)} = \sum_{i=k}^{n} P(i) * level(a_i) + \sum_{i=k}^{n} Q(i) * level(E_i) - 1$$

The C (i, J) can be computed as:

C (i, J) = min {C (i, k-1) + C (k, J) + P (K) + w (i, K-1) + w (K, J)}
i<k<J

= min {C (i, K-1) + C (K, J)} + w (i, J) -- (1)
i<k<J

Where W (i, J) = P (J) + Q (J) + w (i, J-1) -- (2)

Initially C (i, i) = 0 and w (i, i) = Q (i) for 0 ≤ i ≤ n.

Equation (1) may be solved for C (0, n) by first computing all C (i, J) such that J - i = 1 Next, we can compute all C (i, J) such that J - i = 2, Then all C (i, J) with J - i = 3
and so on.
C (i, J) is the cost of the optimal binary search tree 'Tij' during computation we record the root R (i, J) of each tree 'Tij'. Then an optimal binary search tree may be constructed from these R (i, J). R (i, J) is the value of 'K' that minimizes equation (1).

We solve the problem by knowing W (i, i+1), C (i, i+1) and R (i, i+1), 0 ≤ i < 4;

Knowing W (i, i+2), C (i, i+2) and R (i, i+2), 0 ≤ i < 3 and repeating until W (0, n), C (0, n) and R (0, n) are obtained.

The results are tabulated to recover the actual tree.

Example 1:

Let n = 4, and (a1, a2, a3, a4) = (do, if, need, while) Let P (1: 4) = (3, 3, 1, 1) and Q (0: 4) = (2, 3, 1, 1, 1)

Solution:

Table for recording W (i, j), C (i, j) and R (i, j):

Column Row	**0**	**1**	**2**	**3**	**4**
0	2,0,0	3,0,0	1, 0,0	1, 0, 0,	1, 0, 0
1	8,8,1	7,7,2	3, 3,3	3, 3, 4	
2	12, 19, 1	9, 12, 2	5, 8,3		
3	14, 25, 2	11, 19, 2			
4	16, 32, 2				

This computation is carried out row-wise from row 0 to row 4. Initially, W (i, i) = Q (i) and C (i, i) = 0 and R (i, i) = 0, 0 ≤ i < 4.

Solving for C (0, n):

First, computing all C (i, j) such that j - i = 1; j = i + 1 and as $0 \leq i < 4$; i = 0, 1, 2 and 3; $i < k \leq$ J. Start with i = 0; so j = 1; as $i < k \leq j$, so the possible value for k = 1

W(0,1)=P(1)+Q(1)+W(0,0)=3+3+2=8 C (0, 1) = W
(0, 1) + min {C (0, 0) + C (1, 1)} = 8
R (0, 1) = 1 (value of 'K' that is minimum in the above equation). Next with i

= 1; so j = 2; as $i < k \leq j$, so the possible value for k = 2

W (1, 2)=P(2)+Q(2)+W(1, 1) =3+1 +3=7
C (1, 2)= W (1, 2) + min {C (1, 1) +C(2, 2)}=7
R (1, 2)= 2

Next with i = 2; so j = 3; as $i < k \leq j$, so the possible value for k = 3

W (2,3) =P(3)+Q(3)+W(2, 2)=1+1 +1= 3

C (2, 3) = W (2, 3) + min {C (2, 2)+C(3, 3)}=3+[(0 +0)] =3
ft (2, 3) = 3

Next with i = 3; so j = 4; as $i < k \leq j$, so the possible value for k = 4

W (3,4) =P(4)+Q(4)+W(3,3) =1+1 +1=3
C (3, 4) = W (3, 4) + min {[C (3, 3) +C(4, 4)]}=3 +[(0 +0)] =3
ft (3, 4) = 4

Second, Computing all C (i, j) such that j - i = 2; j = i + 2 and as $0 \leq i < 3$; i = 0, 1, 2; $i < k \leq J$.
Start with i = 0; so j = 2; as $i < k \leq J$, so the possible values for k = 1 and 2.

W(0,2)=P(2)+Q(2)+W(0,1)=3+1+8=12
C (0, 2) = W (0, 2) + min {(C (0, 0) + C (1, 2)), (C (0, 1) + C (2, 2))} = 12
+ min {(0 + 7, 8 + 0)} = 19
ft (0, 2) = 1
Next, with i = 1; so j = 3; as $i < k \leq j$, so the possible value for k = 2 and 3.

W (1, 3)=P(3) +Q(3)+W(1,2)=1+1+7=9

C (1, 3)=W(1, 3) + min {[C (1, 1) + C (2, 3)], [C (1, 3) 2) + C (3, 3)]}
= W (1, + min {(0 + 3), (7 + 0)} = 9 + 3 = 12

ft (1, 3) = 2

Next, with i = 2; so j = 4; as $i < k \leq j$, so the possible value for k = 3 and 4.

W(2,4)=P(4)+Q(4)+W(2,3)=1+1+3=5
C (2, 4) = W (2, 4) + min {[C (2, 2) + C (3, 4)], [C (2, 3) + C (4, 4)]
= 5 + min {(0 + 3), (3 + 0)} = 5 + 3 =
8 ft (2, 4) = 3

Third, Computing all C (i, j) such that J - i = 3; j = i + 3 and as $0 < i \leq 2$; i = 0, 1; $i < k \leq J$.
Start with i = 0; so j = 3; as $i < k \leq j$, so the possible values for k = 1, 2 and 3.

W(0, 3) =P(3)+Q(3)+W(0,2)=1+1=+12=14
C (0, 3) W (0, 3) + min {[C (0, 0) + C (1, 3)], [C (0, 1) + C (2, 3)],
[C (0, 2) + C (3, = 3)]}

+ 0)} = 14

ft (0, 3) 14 + min {(0 + 12), (8 + 3), (19 = 2 +11=25

Start with i = 1; so j = 4; as $i < k \leq j$, so the possible values for k = 2, 3 and 4.

W(1, 4) =P(4)+Q(4)+W(1,3)=1+1+9=11=W 2) 4)],

C (1, 4) (1, 4) + min {[C (1, 1) + C (2, 4)], [C (1, + C (3,
[C (1, 3) + C (4, 4)]}

= 11 + min {(0 + 8), (7 + 3), (12 + 0)} = 11 = 2 +8=19

ft (1, 4)

Fourth, Computing all C (i, j) such that j - i = 4; j = i + 4 and as 0 ≤ i < 1; i = 0; i < k ≤ J.

Start with i = 0; so j = 4; as i < k ≤ j, so the possible values for k = 1, 2, 3 and 4.

W (0, 4) =P(4) +Q(4)+W(0,3) =1+1 +14=16

C (0, 4) = W(0, 4) + min {[C (0, 0) +C(1, 4)], [C (0, 1) + C (2, 4)],
[C (0, 2) +C(3, 4)], [C (0, 3) + C (4, 4)]}

= 16 + min [0 + 19, 8 + 8, 19+3, 25+0] = 16 + 16 = 32 ft (0, 4)=2

From the table we see that C (0, 4) = 32 is the minimum cost of a binary search tree for (a1, a2, a3, a4). The root of the tree 'T04' is 'a2'.

Hence the left sub tree is 'T01' and right sub tree is T24. The root of 'T01' is 'a1' and the root of 'T24' is a3.

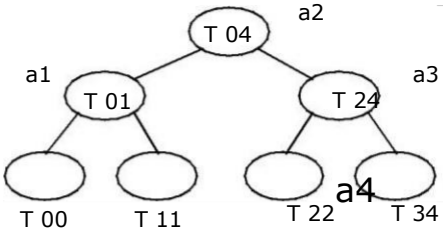
The left and right sub trees for 'T01' are 'T00' and 'T11' respectively. The root of T01 is 'a1'

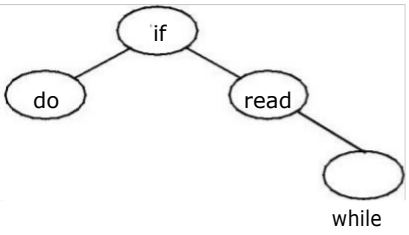
The left and right sub trees for T24 are T22 and T34 respectively. The

root of T24 is 'a3'.

The root of T22 is null The

root of T34 is a4.





Example 2:

Consider four elements a1, a2, a3 and a4 with Q0 = 1/8, Q1 = 3/16, Q2 = Q3 = Q4 = 1/16 and p1 = 1/4, p2 = 1/8, p3 = p4 =1/16. Construct an optimal binary search tree. Solving for C (0, n):

First, computing all C (i, j) such that j - i = 1; j = i + 1 and as 0 ≤ i < 4; i = 0, 1, 2 and 3; i < k ≤ J. Start with i = 0; so j = 1; as i < k ≤ j, so the possible value for k = 1 W (0, 1)

=P(1)+Q(1)+W(0,0)=4+3+2=9

C (0, 1) = W (0, 1) + min {C (0, 0) + C (1, 1)} = 9 + [(0 + 0)] = 9 ft (0, 1)
= 1 (value of 'K' that is minimum in the above equation).

Next with i = 1; so j = 2; as i < k ≤ j, so the possible value for k = 2 W (1, 2)

=P(2)+Q(2)+W(1,1)=2+1+3=6
C (1, 2) = W (1, 2) + min {C (1, 1) + C (2, 2)} = 6 + [(0 + 0)] = 6 ft
(1, 2)=2

Next with i = 2; so j = 3; as i < k ≤ j, so the possible value for k = 3

+1=3
3)} = 3W+ (2,[(03)+ 0)]=P=(3) + Q (3) + W (2, 2) =
1 + 1 C (2, 3) = W (2, 3) + min {C (2, 2) + C (3,

ft (2, 3) = 3

Next with i = 3; so j = 4; as i < k ≤ j, so the possible value for k = 4

W (3, 4) =P(4)+Q(4)+W(3,3) =1+1 +1=3
C (3, 4) = W (3, 4) + min {[C (3, 3) + C (4, 4)]} =3 +[(0 +0)] =3
ft (3, 4) = 4

Second, Computing all C (i, j) such that j - i = 2; j = i + 2 and as 0 ≤ i < 3; i = 0, 1, 2; i < k ≤ J Start with i = 0; so j = 2; as i < k ≤ j, so the possible values for k = 1 and 2. W (0, 2)

=P(2)+Q(2)+W(0,1)=2+1+9=12

C (0, 2) = W (0, 2) + min {(C (0, 0) + C (1, 2)), (C (0, 1) + C (2, 2))} = 12
+ min {(0 + 6, 9 + 0)} = 12 + 6 = 18
ft (0, 2) = 1
Next, with i = 1; so j = 3; as i < k ≤ j, so the possible value for k = 2 and 3.

W (1, 3)=P(3) +Q(3)+W(1,2)=1+1+6=8
C (1, 3)=W(1, 3) + min {[C (1, 1) + C (2, 3)], [C (1, 2) + C (3, 3)]}
=W(1, 3) + min {(0 + 3), (6 + 0)} = 8 + 3 = 11
ft (1, 3) = 2

Next, with i = 2; so j = 4; as i < k ≤ j, so the possible value for k = 3 and 4. W (2, 4)

=P(4)+Q(4)+W(2,3)=1+1+3=5
C (2, 4) = W (2, 4) + min {[C (2, 2) + C (3, 4)], [C (2, 3) + C (4, 4)]
= 5 + min {(0 + 3), (3 + 0)} = 5 + 3 =
8 ft (2, 4) = 3

Third, Computing all C (i, j) such that J - i = 3; j = i + 3 and as 0 ≤ i < 2; i = 0, 1; i < k ≤ J.
Start with i = 0; so j = 3; as i < k ≤ j, so the possible values for k = 1, 2 and 3.

W(0,3)=P(3)+Q(3)+W(0,2)=1+1+12=14
C (0, 3) = W (0, 3) + min {[C (0, 0) + C (1, 3)], [C (0, 1) + C (2, 3)], [C (0, 2)
+ C (3, 3)]}
= 14 + min {(0 + 11), (9 + 3), (18 + 0)} = 14 + 11 = 25 ft (0,
3)=1

Start with i = 1; so j = 4; as i < k ≤ j, so the possible values for k = 2, 3 and 4.

W (1, 4) =P(4)+Q(4)+W(1,3)=1+1+8=10=W 2)
C (1, 4) (1, 4) + min {[C (1, 1) + C (2, 4)], [C (1, +C(3, 4)],
[C (1, 3) + C (4, 4)]}
= 10 + min {(0 + 8), (6 + 3), (11 + 0)} = 10 = 2 +8=18
ft (1, 4)

Fourth, Computing all C (i, j) such that J - i = 4; j = i + 4 and as 0 ≤ i < 1; i = 0;
i < k ≤ J. Start with i = 0; so j = 4; as i < k ≤ j, so the possible values for k = 1, 2, 3 and 4.

W (0, 4) =P(4) +Q(4)+W(0,3) =1+1+14=16
C (0, 4) = W (0, 4) + min {[C (0, 0) + C (1, 4)], [C (0, 1) + C (2, 4)],
[C (0, 2) + C (3, 4)], [C (0, 3) + C (4, 4)]}

= 16 + min [0 + 18, 9 + 8, 18 + 3, 25 + 0] = 16 + 17 = 33 R (0, 4)
= 2

Table for recording W (i, j), C (i, j) and R (i, j)

Column Row	**0**	**1**	**2**	**3**	**4**
0	2,0,0	1,0,0	1,0,0	1, 0, 0,	1, 0, 0
1	9,9,1	6,6,2	3,3,3	3, 3, 4	
2	12, 18, 1	8, 11, 2	5,8,3		
3	14, 25, 2	11, 18, 2			
4	16, 33, 2				

From the table we see that C (0, 4) = 33 is the minimum cost of a binary search tree for (a1, a2, a3, a4)

The root of the tree 'T04' is 'a2'.

Hence the left sub tree is 'T01' and right sub tree is T24. The root of 'T01' is 'a1' and the root of 'T24' is a3.

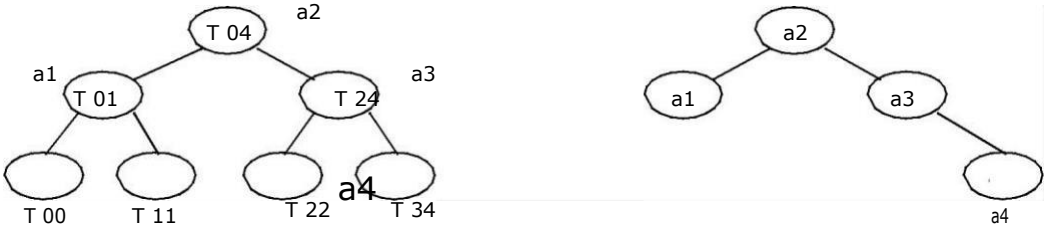
The left and right sub trees for 'T01' are 'T00' and 'T11' respectively. The root of T01 is 'a1'

The left and right sub trees for T24 are T22 and T34 respectively. The

root of T24 is 'a3'.

The root of T22 is null. The

root of T34 is a4.



0/1 – KNAPSACK

We are given n objects and a knapsack. Each object i has a positive weight wi and a positive value Vi. The knapsack can carry a weight not exceeding W. Fill the knapsack so that the value of objects in the knapsack is optimized.

A solution to the knapsack problem can be obtained by making a sequence of decisions on the variables x1, x2, ... , xn. A decision on variable xi involves determining which of the values 0 or 1 is to be assigned to it. Let us assume that

decisions on the xi are made in the order xn, xn-1, ... x1. Following a decision on xn, we may be in one of two possible states: the capacity remaining in m – wn and a profit of pn has accrued. It is clear that the remaining decisions xn-1,..... , x1 must be optimal with respect to the problem state resulting from the decision on xn. Otherwise, xn,. . . . , x1 will not be optimal. Hence, the principal of optimality holds.

Fn (m) = max {fn-1 (m), fn-1 (m - wn) + pn} -- 1

For arbitrary fi (y), i > 0, this equation generalizes to:

Fi (y) = max {fi-1 (y), fi-1 (y - wi) + pi} -- 2

Equation-2 can be solved for fn (m) by beginning with the knowledge fo (y) = 0 for all y and fi (y) = - ~, y < 0. Then f1, f2, . . . fn can be successively computed using equation–2.

When the wi's are integer, we need to compute fi (y) for integer y, $0 \leq y \leq m$. Since fi (y) = - ~ for y < 0, these function values need not be computed explicitly. Since each fi can be computed from fi - 1 in **Θ** (m) time, it takes **Θ** (m n) time to compute fn. When the wi's are real numbers, fi (y) is needed for real numbers y such that $0 < y \leq m$. So, fi cannot be explicitly computed for all y in this range. Even when the wi's are integer, the explicit **Θ** (m n) computation of fn may not be the most efficient computation. So, we explore **an alternative method for both cases.**

The fi (y) is an ascending step function; i.e., there are a finite number of y's, 0 = y1 < y2 < . . . . < yk, such that fi (y1) < fi (y2) < ..... < fi (yk); fi (y) = - ~ , y < y1; fi (y) = f (yk), y ≥ yk; and fi (y) = fi (yj), $yj \leq y \leq yj+1$. So, we need to compute only fi (yj), $1 \leq j < k$. We use the ordered set $S^i = \{(f (yj), yj) \mid 1 \leq j \leq k\}$ to represent fi (y). Each number of S^i is a pair (P, W), where P = fi (yj) and W = yj. Notice that $S^0 = \{(0, 0)\}$. We can compute S^{i+1} from Si by first computing:

Si 1 = {(P, W) | (P – pi, W – wi) e S^i}

Now, S^{i+1} can be computed by merging the pairs in S^i and Si 1 together. Note that if Si+1 contains two pairs (Pj, Wj) and (Pk, Wk) with the property that $Pj \leq Pk$ and Wj > Wk, then the pair (Pj, Wj) can be discarded because of equation-2. Discarding or purging rules such as this one are also known as dominance rules. Dominated tuples get purged. In the above, (Pk, Wk) dominates (Pj, Wj).

Reliability Design

The problem is to design a system that is composed of several devices connected in series. Let ri be the reliability of device Di (that is ri is the probability that device i will function properly) then the reliability of the entire system is fT ri. Even if the individual devices are very reliable (the ri's are very close to one), the reliability of the system may not be very good. For example, if n = 10 and ri = 0.99, i ≤ i ≤ 10, then fT ri = .904. Hence, it is desirable to duplicate devices. Multiply copies of the same device type are connected in parallel.

If stage i contains mi copies of device Di. Then the probability that all mi have a malfunction is (1 - ri) mi. Hence the reliability of stage i becomes $1 - (1 - r_i)^{mi}$.

The reliability of stage 'i' is given by a function ~i (mi).

Our problem is to use device duplication. This maximization is to be carried out under a cost constraint. Let ci be the cost of each unit of device i and let c be the maximum allowable cost of the system being designed.

We wish to solve:

Maximize ~ *qi* (*mi* ~
$1 < i < n$

Subject to ~ *Ci mi* < *C*
$1 < i < n$

mi ≥ 1 and interger, 1 ≤ i ≤ n

Assume each Ci > 0, each mi must be in the range 1 ≤ mi ≤ ui, where

$$ui = \left\lfloor \frac{C + Ci - \sum_{j=1}^{n} Cj}{Ci} \right\rfloor$$

The upper bound ui follows from the observation that mj ≥ 1 An

optimal solution m1, m2 ..... mn is the result of a sequence of decisions, one decision for each mi.

Let fi (x) represent the maximum value of Subject $\prod_{1 < j < i} q_j(m_j)$

to the constrains:

$\sum_{1 < j < i} C_j m_j \sim x$ and $1 \le m_j \le u_j$, $1 \le j \le i$

The last decision made requires one to choose mn from {1, 2, 3, ... un}

Once a value of mn has been chosen, the remaining decisions must be such as to use the remaining funds C – Cn mn in an optimal way.

The principle of optimality holds on

f_n ~C ~ ~max { On (m_n) fn _ 1 (C - C_n
m_n) } 1 < m_n < u_n

for any fi (xi), i > 1, this equation generalizes to

$_n$ (x) = max { ci (mi) f i - 1 (x - Ci mi) } 1 < mi < ui

clearly, f0 (x) = 1 for all x, 0 ≤ x ≤ C and f (x) = -oo for all x < 0. Let S^i consist of tuples of the form (f, x), where f = fi (x).

There is atmost one tuple for each different 'x', that result from a sequence of decisions on m1, m2, ... mn. The dominance rule (f1, x1) dominate (f2, x2) if f1 ≥ f2 and x1 ≤ x2. Hence, dominated tuples can be discarded from S^i.

Example 1:

Design a three stage system with device types D1, D2 and D3. The costs are $30, $15 and $20 respectively. The Cost of the system is to be no more than $105. The reliability of each device is 0.9, 0.8 and 0.5 respectively.

Solution:

We assume that if if stage I has mi devices of type i in parallel, then *0* i (mi) =1 – (1- ri)^{mi} Since,

we can assume each ci > 0, each mi must be in the range 1 ≤ mi ≤ ui. Where:

$$u_i = \left\lfloor C + C_i \right\rfloor - C \Big/ C_i$$

Using the above equation compute u1, u2 and u3.

$$u1 = \frac{105+30-(30+15+20)}{30} = \frac{70}{30} = 2$$

$$u2 = \frac{105+15-(30+15+20)}{15} = \frac{55}{15} = 3$$

$$u3 = \frac{105+20-(30+15+20)}{20} = \frac{60}{20} = 3$$

We use S° -* *i*:*stage number and J*: *no*. *of devices in stage i* = mi

$= \{f_o(x), x\}$ *initially* $f_o(x) = 1$ *and* $x = 0$, *so,* $S^{o} = \{1, 0\}$

Compute S^1, S^2 and S^3 as follows:

S1 = depends on u1 value, as u1 = 2, so

$S1 = \{S1, S1\}$

1 2

S2 = depends on u2 value, as u2 = 3, so

$s2=\{S^2, S^2, S^2\}$

1 2 3

S3 = depends on u3 value, as u3 = 3, so

$S3=\{S^3, S^3, S^3\}$

1 2 3

, 1 *x*

Now find $S(x)$,

$f1(x) = \{01(1) f_o \sim \sim, 01(2) f0()\}$ With devices m1 = 1 and m2 = 2 Compute Ø1 (1) and Ø1 (2) using the formula: $Øi(mi)) = 1 - (1 - ri)^{mi}$

$\sim\sim 1\sim \sim 1\sim\sim 1 \sim r \sim m\ 1 = 1-(1-0.9)^1 = 0.9$

1 1 ~(2) = 1-(1- 0.9) 2 =0.99 1

S ~~1$f1 \sim x\sim, x \sim \sim$

1 ~~0.9 , 30

$S2$

1 =10.99 , 30 + 30 } =(0.99, 60

Therefore, $S^1 = \{(0.9, 30), (0.99, 60)\}$

Next fin d 2 ~ ~ ~ *f*

S 12 (*x*), *x* ~~

$f2(x) = \{02(1) * f1(), 02(2) * f1(), 02(3) * f1()\}$

~2 ~1~ ~ 1 ~ ~1 ~ *rI* ~ = 1 – (1 – 0.8) = 1 – 0.2 = 0.8
mi 1

~~2~ ~ 1 ~~1 ~ 0.8~ 2 = 0.96 2

*0*2(3) = 1 - (1 - 0.8) 3 = 0.992

= {(0.8(0.9),30 + 15), (0.8(0.99),60 + 15)} = {(0.72, 45), (0.792, 75)} =
{(0.96(0.9),30 + 15 +15) , (0.96(0.99),60 + 15 + 15)}
= {(0.864, 60), (0.9504, 90)}

= {(0.992(0.9),30 + 15 +15+15) , (0.992(0.99),60 + 15 + 15+15)}
= {(0.8928, 75), (0.98208, 105)}
*S*2={*S*^{2}, *S*^{2}, *S*^{2}}
1 2 3

By applying Dominance rule to S^2:
Therefore, S2 = {(0.72, 45), (0.864, 60), (0.8928, 75)} Dominance Rule:

If S^i contains two pairs (f1, x1) and (f2, x2) with the property that f1 ≥ f2 and x1 ≤ x2, then (f1, x1) dominates (f2, x2), hence by dominance rule (f2, x2) can be discarded. Discarding or pruning rules such as the one above is known as dominance rule. Dominating tuples will be present in S^i and Dominated tuples has to be discarded from Si.

Case 1: if f1 ≤ f2 and x1 > x2 then discard (f1, x1) Case 2: if f1 ≥

f2 and x1 < x2 the discard (f2, x2) Case 3: otherwise simply write

(f1, x1)

S2 = {(0.72, 45), (0.864, 60), (0.8928, 75)}

Ø 3 (1) = 1 ~ ~1 _ *rI* ~ *mi* = 1 – (1 – 0.5)^1 = 1 – 0.5 = 0.5
Ø 2~2~~1~~1=0.75
~ *S* 0.5~ 2
3 = 0.875

Ø *S* ^{2} ~3~~1~ ~1
~ 0.5~ 3
3
*S*3 ¬
2
S 13 = {(0.5 (0.72), 45 + 20), (0.5 (0.864), 60 + 20), (0.5 (0.8928), 75 + 20)}

S 13 = {(0.36, 65), (0.437, 80), (0.4464, 95)}

S_2^3 ={(0.75 (0.72), 45 + 20 + 20), (0.75 (0.864), 60 + 20 +20),
(0.75 (0.8928), 75 + 20 + 20)}

= {(0.54, 85), (0.648, 100), (0.6696, 115)}

S 3 3 = { 0.875 (0.72), 45 + 20 + 20 + 20), 0.875 (0.864),60 + 20 + 20 + 20,
0.875 (0.8928), 75 + 20 + 20 + 20 }

S 3

3 = {(0.63, 105), (1.756, 120), (0.7812, 135)}
If cost exceeds 105, remove that tuples

S3 = {(0.36, 65), (0.437, 80), (0.54, 85), (0.648, 100)}

The best design has a reliability of 0.648 and a cost of 100. Tracing back for the solution through S^i 's we can determine that m3 = 2, m2 = 2 and m1 = 1.

Other Solution:

According to the principle of optimality:

fn(C) = max {~n (mn). fn-1 (C - Cn mn) with fo (x) = 1 and 0 ≤ x ≤ C; 1 ~ *mn* < *un*

Since, we can assume each ci > 0, each mi must be in the range 1 ≤ mi ≤ ui. Where:

S2 ={(0.75 (0.72), 45 + 20 + 20), (0.75 (0.864), 60 + ~ *u* (*n* ~ ~
20 +20),*i*

= ~ i*C* + *Ci* _ ~*CJ* r / *Ci* I ~

~ *i* ~ ~~

Using the above equation compute u1, u2 and u3.

$$u1 = \frac{105 - 30 + \frac{70}{30}}{30} = 2$$

$$u2 = \frac{105 - 1530 + \frac{55}{15}}{30} = 3$$

$$u3 = \frac{105 - 15 + 20 - \frac{60}{20}}{20} = 3$$

f3 (105) = max {~3 (m3). f2 (105 - 20m3)} 1 < *m*3 ! *u*3

= max {3(1) f2(105 - 20), 63(2) f2(105 - 20x2), ~3(3) f2(105 -20x3)} = max

{0.5 f2(85), 0.75 f2(65), 0.875 f2(45)}

= max {0.5 x 0.8928, 0.75 x 0.864, 0.875 x 0.72} = 0.648.

= max {2 (m2). f1 (85 -15m2)}
1 ! *m*2 ! *u*2

f2 $^{(85)}$ = max {2(1).f1(85 - 15), ~2(2).f1(85 - 15x2), ~2(3).f1(85 - 15x3)} = max {0.8 f1(70), 0.96 f1(55), 0.992 f1(40)}

= max {0.8 x 0.99, 0.96 x 0.9, 0.99 x 0.9} =

0.8928 f1 (70) = max {~1(m1). f0(70 - 30m1)}
1 ! *m*1 ! *u*1

= max {~1(1) f0(70 - 30), t1(2) f0 (70 - 30x2)}

= max {~1(1) x 1, $_{t1(2)}$ x 1} = max {0.9, 0.99} = 0.99

f1 (55) = max {t1(m1). f0(55 - 30m1)}

1 ! *m*1 ! *u*1

= max {~1(1) f0(50 - 30), t1(2) f0(50 - 30x2)}

= max {~1(1) x 1, $_{t1(2)}$ x -oo} = max {0.9, -oo} = 0.9

f1 (40) = max {~1(m1). f0 (40 - 30m1)}

1 ! *m*1 ! *u*1

= max {~1(1) f0(40 - 30), t1(2) f0(40 - 30x2)}

= max {~1(1) x 1, $_{t1(2)}$ x -oo} = max{0.9, -oo} = 0.9

(65) = max {2(m2). f1(65 -15m2)} 1 ! *m*2 ! *u*2

= max {2(1) f1(65 - 15), 62(2) f1(65 - 15x2), ~2(3) f1(65 - 15x3)} = max {0.8 f1(50), 0.96 f1(35), 0.992 f1(20)}

= max {0.8 x 0.9, 0.96 x 0.9, -oo} = 0.864

(50) = max {~1(m1). f0(50 - 30m1)} 1 ! *m*1 ! *u*1

= max {~1(1) f0(50 - 30), t1(2) f0(50 - 30x2)}

= max {~1(1) x 1, $_{t1(2)}$ x -oo} = max{0.9, -oo} = 0.9 f1 (35)

= max ~1(m1). f0(35 - 30m1)}

1 ! *m*1 ! *u*1

= max {~1(1).f0(35-30), ~1(2).f0(35-30x2)}

= max {~1(1) x 1, $_{t1(2)}$ x -oo} = max{0.9, -oo} = 0.9

(20) = max {~1(m1). f0(20 - 30m1)} 1 ! *m*1 ! *u*1

= max {~1(1) f0(20 - 30), t1(2) f0(20 - 30x2)}

= max {~1(1) x -, ~1(2) x -oo} = max{-oo, -oo} = -oo f2

(45) = max {2(m2). f1(45 -15m2)}

1 ! *m*2 ! *u*2

= max {2(1) f1(45 - 15), ~2(2) f1(45 - 15x2), ~2(3) f1(45 - 15x3)} = max {0.8 f1(30), 0.96 f1(15), 0.992 f1(0)}

= max {0.8 x 0.9, 0.96 x -, 0.99 x -oo} = 0.72

f1 (30) = max {~1(m1). f0(30 - 30m1)} 1<*m*1~*u*1

= max {~1(1) f0(30 - 30), t1(2) f0(30 - 30x2)}

= max {~1(1) x 1, t1(2) x -oo} = max{0.9, -oo} = 0.9 Similarly, f1 (15) = -, f1 (0)

= -.

The best design has a reliability = 0.648 and Cost
= 30 x 1 + 15 x 2 + 20 x 2 = 100.

Tracing back for the solution through S^i 's we can determine that: m3 = 2, m2 = 2 and

m1 = 1.

MODULE IV:

Backtracking: General method, applications-n-queen problem, sum of subsets problem, graph coloring, Hamiltonian cycles.

Branch and Bound: General method, applications - Travelling sales person problem,0/1 knapsack problem- LC Branch and Bound solution, FIFO Branch and Bound solution.

Backtracking (General method)

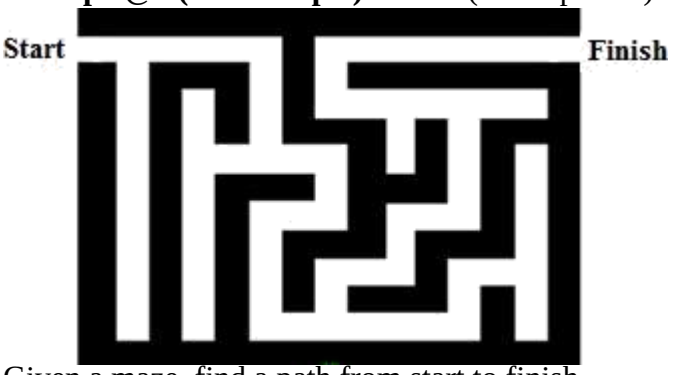
Many problems are difficult to solve algorithmically. Backtracking makes it possible to solve at least some large instances of difficult combinatorial problems.

Suppose you have to make a series of decisions among various choices, where

- You don't have enough information to know what to choose
- Each decision leads to a new set of choices.
- Some sequence of choices (more than one choices) may be a solution to your problem.

Backtracking is a methodical (Logical) way of trying out various sequences of decisions, until you find one that "works"

Example@1 (net example) : Maze (a tour puzzle)



- Given a maze, find a path from start to finish.

In maze, at each intersection, you have to decide between 3 or fewer choices: ✓

Go straight

✓ Go left ✓

Go right

- You don't have enough information to choose correctly
- Each choice leads to another set of choices.
- One or more sequences of choices may or may not lead to a solution.
- Many types of maze problem can be solved with backtracking.

Example@ 2 (text book):

Sorting the array of integers in a[1:n] is a problem whose solution is expressible by an n-tuple x_i $\rightarrow$ is the index in 'a' of the i^{th} smallest element.

The criterion function 'P' is the inequality $a[x_i] \leq a[x_{i+1}]$ for $1 \leq i \leq n$ S_i $\rightarrow$ is finite and includes the integers 1 through n.

m_i $\rightarrow$ size of set S_i

$m = m_1 m_2 m_3 \text{---} m_n$ n tuples that possible candidates for satisfying the function P.

With brute force approach would be to form all these n-tuples, evaluate (judge) each one with P and save those which yield the optimum.

By using backtrack algorithm; yield the same answer with far fewer than 'm' trails. Many of the problems we solve using backtracking requires that all the solutions satisfy a complex set of constraints.

For any problem these constraints can be divided into two categories:

Explicit constraints.

Implicit constraints.

Explicit constraints: Explicit constraints are rules that restrict each x_i to take on values only from a given set.

Example: $x_i \geq 0$ or si={all non negative real numbers}

X_i=0 or 1 or S_i={0, 1}

$l_i \leq x_i \leq u_i$ or s_i={a: $l_i \leq a \leq u_i$ }

The explicit constraint depends on the particular instance I of the problem being solved.

All tuples that satisfy the explicit constraints define a possible solution space for I.

Implicit Constraints:

The implicit constraints are rules that determine which of the tuples in the solution space of I satisfy the criterion function. Thus implicit constraints describe the way in which the X_i must relate to each other.

Applications of Backtracking:

N Queens Problem

Sum of subsets problem

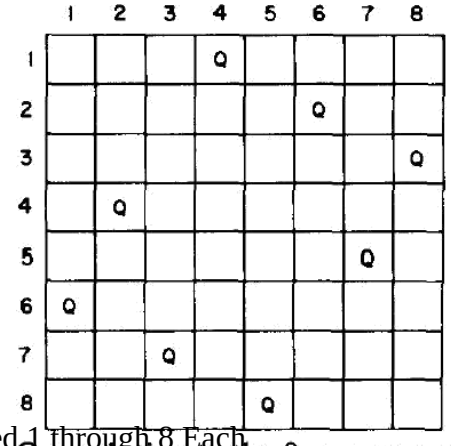
Graph coloring

Hamiltonian cycles.

N-Queens Problem:

It is a classic combinatorial problem. The eight queen's puzzle is the problem of placing eight queens puzzle is the problem of placing eight queens on an 8×8 chessboard so that no two queens attack each other. That is so that no two of them are on the same row, column, or diagonal.

The 8-queens puzzle is an example of the more general n-queens problem of placing n queens on an n×n chessboard.



One solution to the 8-queens problem

Here queens can also be numbered 1 through 8 Each queen must be on a different row Assume queen 'i' is to be placed on row 'i'

All solutions to the 8-queens problem can therefore be represented a s s-tuples(x_1, x_2, x_3—x_8) xi $\rightarrow$ the column on which queen 'i' is placed

si $\rightarrow$ {1, 2, 3, 4, 5, 6, 7, 8}, $1 \leq i \leq 8$

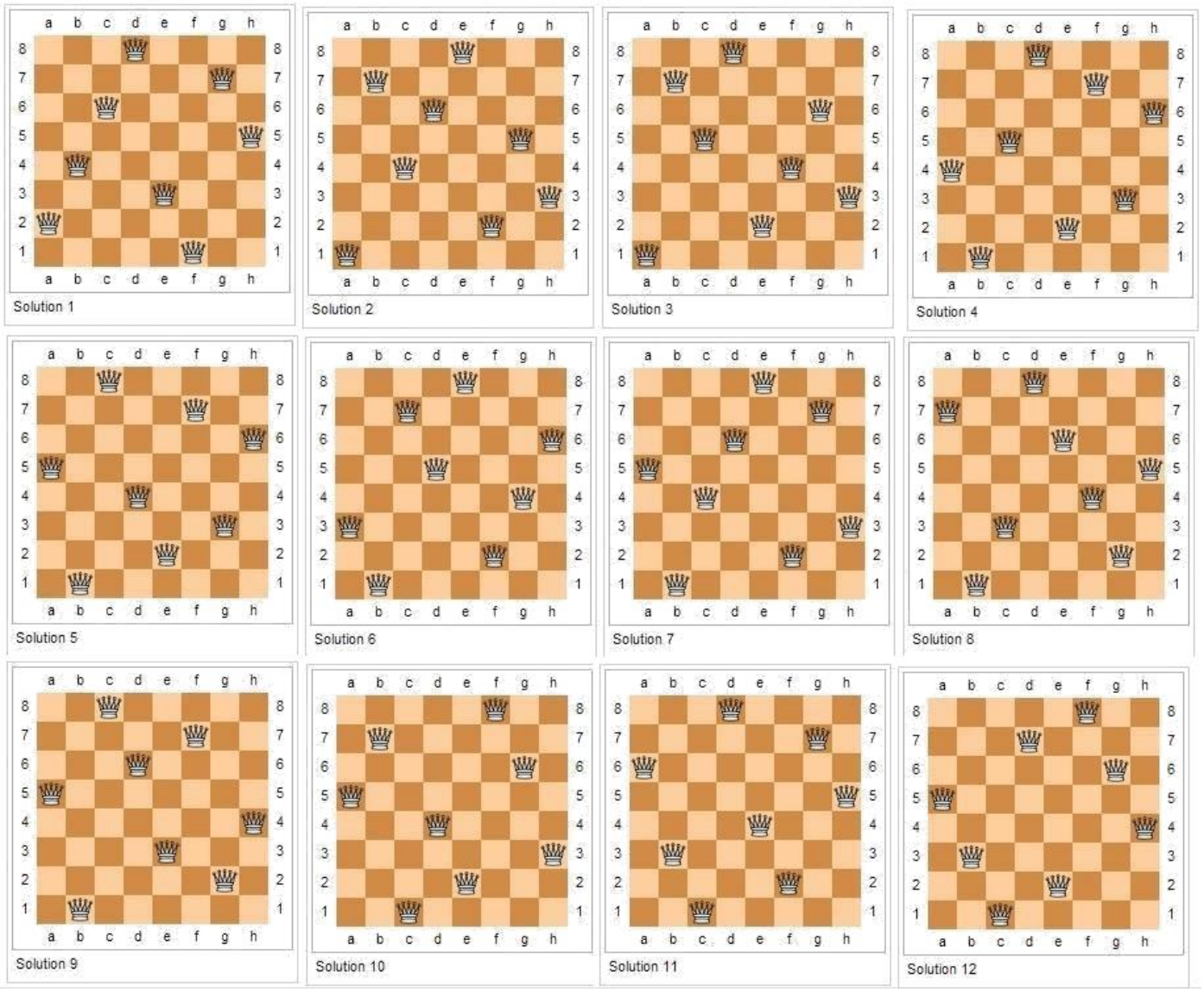
Therefore the solution space consists of 8^8 s-tuples.

The implicit constraints for this problem are that no two x_i's can be the same column and no two queens can be on the same diagonal.

By these two constraints the size of solution pace reduces from 88 tuples to 8!

Tuples. Form example s_i(4,6,8,2,7,1,3,5)

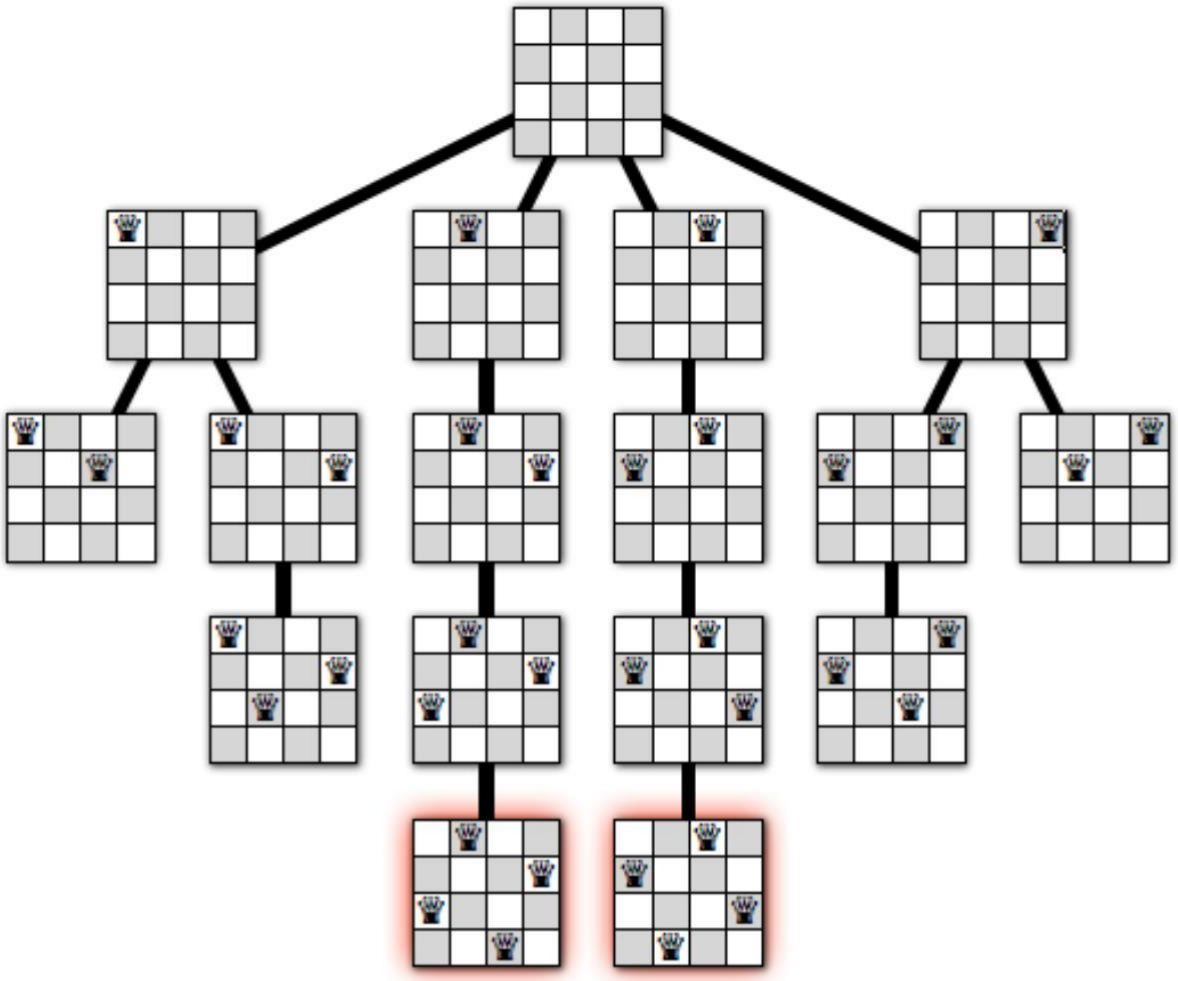
In the same way for n-queens are to be placed on an n×n chessboard, the solution space consists of all n! Permutations of n-tuples (1,2, ---------- n).



Some solution to the 8-Queens problem

Algorithm for new queen be placed	**All solutions to the n·queens problem**

Algorithm Place(k,i)
//Return true if a queen can be placed in kth row & ith column
//Other wise return false
{
for j:=1 to k-1 do
if(x[j]=i or Abs(x[j]-i)=Abs(j-k)))
then return false
return true
}
	Algorithm NQueens(k, n)
	// its prints all possible placements of n-queens on an n×n chessboard.
	{
	for i:=1 to n do{
	if Place(k,i) then
	{ X[k]:=I;
	if(k==n) then write (x[1:n]);
	else NQueens(k+1, n);
	}
	}}



The complete recursion tree for our algorithm for the 4 queens problem.

Sum of Subsets Problem:

Given positive numbers w_i $1 \le i \le n$, & m, here sum of subsets problem is finding all subsets of w_i whose sums are m.

Definition: Given n distinct +ve numbers (usually called weights), desire (want) to find all combinations of these numbers whose sums are m. this is called sum of subsets problem. To formulate this problem by using either fixed sized tuples or variable sized tuples.
Backtracking solution uses the fixed size tuple strategy.

For example:
If n=4 (w_1, w_2, w_3, w_4)=(11,13,24,7) and m=31.
Then desired subsets are (11, 13, 7) & (24, 7).
The two solutions are described by the vectors (1, 2, 4) and (3, 4).

In general all solution are k-tuples (x_1, x_2, x_3---x_k) $1 \le k \le n$, different solutions may have different sized tuples.

- Explicit constraints requires $x_i \in \{j / j \text{ is an integer } 1 \le j \le n\}$
- Implicit constraints requires:

No two be the same & that the sum of the corresponding w_i's be m
i.e., (1, 2, 4) & (1, 4, 2) represents the same. Another constraint is $x_i < x_{i+1}$ $1 \le i \le k$

→

W_i ~~weight of item i~~

M → Capacity of bag (subset)

X_i → the element of the solution vector is either one or zero.

X_i value depending on whether the weight wi is included or not. If
X_i=1 then wi is chosen.
If X_i=0 then wi is not chosen.

$$\underbrace{\sum_{i=1}^{k} W(i)X(i)}_{\text{Total sum till now}} + \underbrace{\sum_{i=k+1}^{n} W(i)}_{\text{Still there}} \geq M$$

The above equation specify that x_1, x_2, x_3, --- x_k cannot lead to an answer node if this condition is not satisfied.

$$\sum_{i=1}^{k} W(i)X(i) + W(k+1) > M$$

The equation cannot lead to solution.

$$B_k(X(1), \ldots, X(k)) = true \text{ iff } \left(\sum_{i=1}^{k} W(i)X(i) + \sum_{i=k+1}^{n} W(i) \geq M \text{ and } \sum_{i=1}^{k} W(i)X(i) + W(k+1) \leq M \right)$$

$$s = \sum_{j=1}^{k-1} W(j)X(j) \text{ and } r = \sum_{j=k}^{n} W(j)$$

Recursive backtracking algorithm for sum of subsets problem Algorithm SumOfSub(s, k, r) {

$$//s = \sum_{j=1}^{k-1} W(j)X(j) \text{ and } r = \sum_{j=k}^{n} W(j)$$

X[k]=1
If(S+w[k]=m) then write(x[1:]); // subset found.
Else if (S+w[k] + w{k+1] ≤ M)
Then SumOfSub(S+w[k], k+1, r-w[k]);
if ((S+r - w{k] ≥ M) and (S+w[k+1] ≤M)) then
{
X[k]=0;
SumOfSub(S, k+1, r-w[k]);
}
}

Graph Coloring:

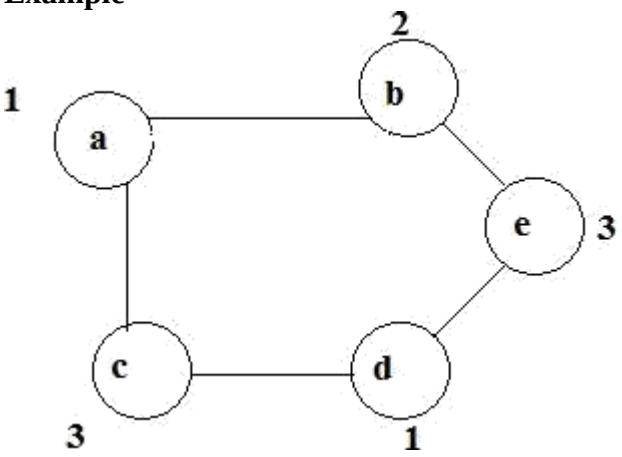
Let G be a undirected graph and 'm' be a given +ve integer. The graph coloring problem is assigning colors to the vertices of an undirected graph with the restriction that no two adjacent vertices are assigned the same color yet only 'm' colors are used.

The optimization version calls for coloring a graph using the minimum number of coloring. The decision version, known as K-coloring asks whether a graph is colourable using at most k- colors.

Note that, if 'd' is the degree of the given graph then it can be colored with 'd+1' colors.

The m- colorability optimization problem asks for the smallest integer 'm' for which the graph G can be colored. This integer is referred as "**Chromatic number**" of the graph.

Example



Above graph can be colored with 3 colors 1, 2, & 3.

The color of each node is indicated next to it.

3-colors are needed to color this graph and hence this graph' Chromatic Number is 3.

A graph is said to be planar iff it can be drawn in a plane (flat) in such a way that no two edges cross each other.

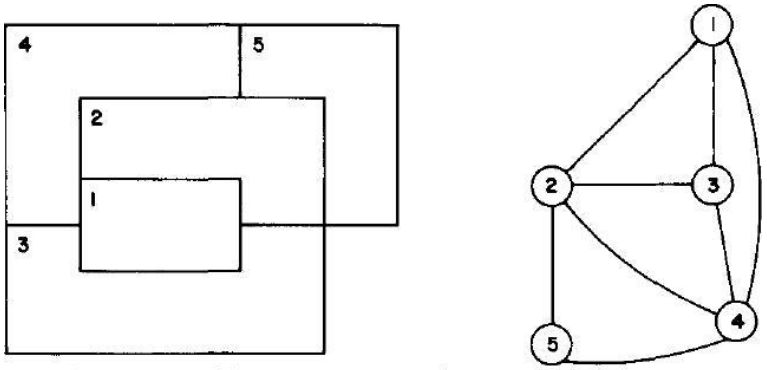
M-Colorability decision problem is the 4-color problem for planar graphs.

Given any map, can the regions be colored in such a way that no two adjacent regions have the same color yet only 4-colors are needed?

To solve this problem, graphs are very useful, because a map can easily be transformed into a graph.

Each region of the map becomes a node, and if two regions are adjacent, then the corresponding nodes are joined by an edge.

- **Example:**



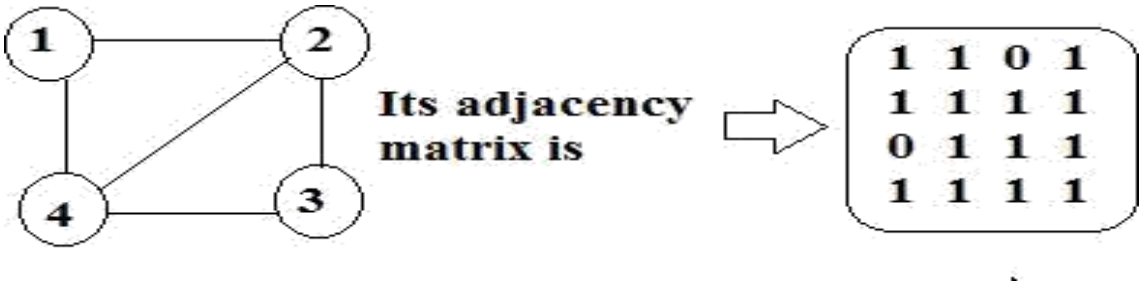
- **A map and its planar graph representation** The above map requires 4 colors.

Many years, it was known that 5-colors were required to color this map.

After several hundred years, this problem was solved by a group of mathematicians with the help of a computer. They show that 4-colors are sufficient.

Suppose we represent a graph by its adjacency matrix G[1:n, 1:n]

Ex:



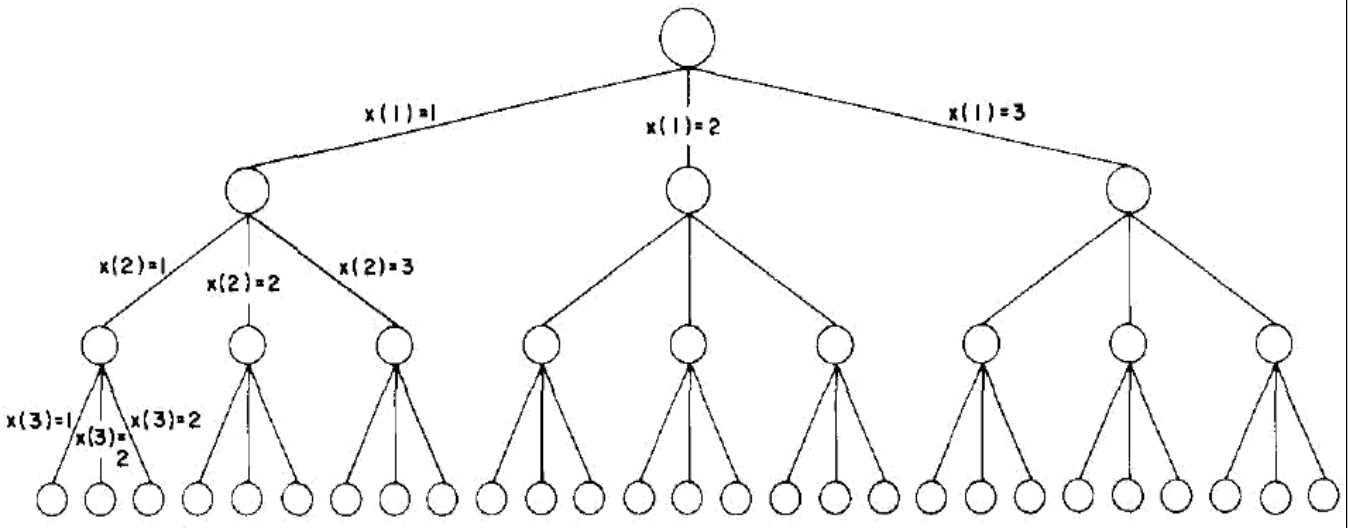
Here G[i, j]=1 if (i, j) is an edge of G, and G[i, j]=0 otherwise.
Colors are represented by the integers 1, 2,---m and the solutions are given by the n-tuple (x1, x2,-- -xn)

$\vec{}$

xi Color of node i.

n=3 $\rightarrow$ nodes

m=3 $\longrightarrow$ colors



State space tree for MCOLORING when $n = 3$ and $m = 3$

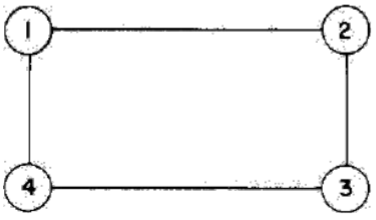
1st node coloured in 3-ways 2nd node coloured in 3-ways 3rd node coloured in 3-ways

So we can colour in the graph in 27 possibilities of colouring.

Finding all m-coloring of a graph	Getting next color

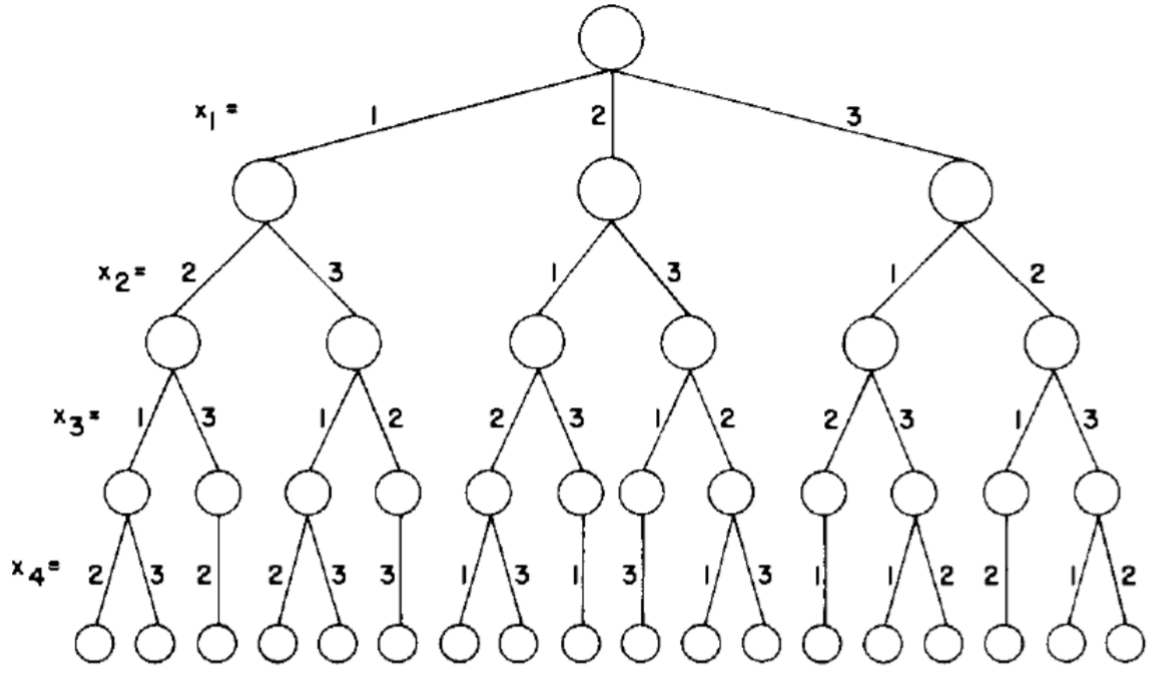
Algorithm mColoring(k){
// g(1:n, 1:n) → boolean adjacency matrix.
// k → index (node) of the next vertex to color.
repeat{
nextvalue(k); // assign to x[k] a legal color.
if(x[k]=0) then return; // no new color possible
if(k=n) then write(x[1: n];
else mcoloring(k+1);
}
until(false)
}
	Algorithm NextValue(k){
	//x[1],x[2],---x[k-1] have been assigned integer values in the range [1, m]
	repeat {
	x[k]=(x[k]+1)mod (m+1); //next highest color
	if(x[k]=0) then return; // all colors have been used.
	for j=1 to n do
	{
	if ((g[k,j]≠0) and (x[k]=x[j]))
	then break;
	}
	if(j=n+1) then return; //new color found
	} until(false)
	}

Previous paper example:



Adjacency matrix is

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$



A 4 node graph and all possible 3 colorings

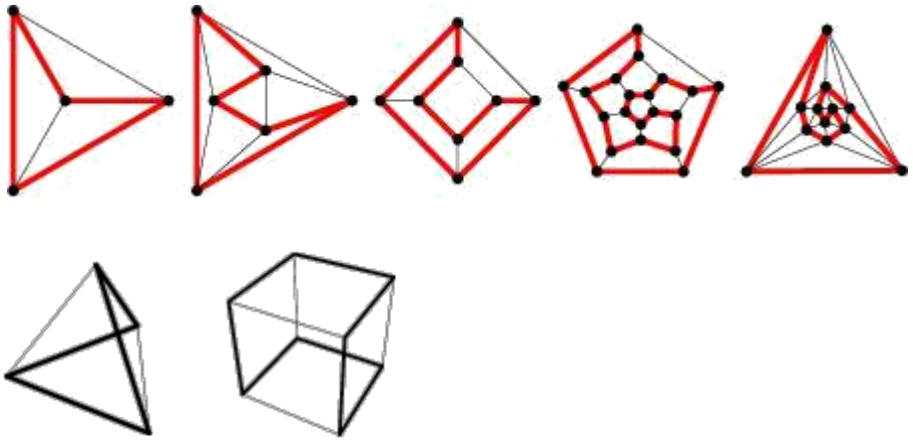
Hamiltonian Cycles:

Def: Let G=(V, E) be a connected graph with n vertices. A Hamiltonian cycle is a round trip path along n-edges of G that visits every vertex once & returns to its starting position.

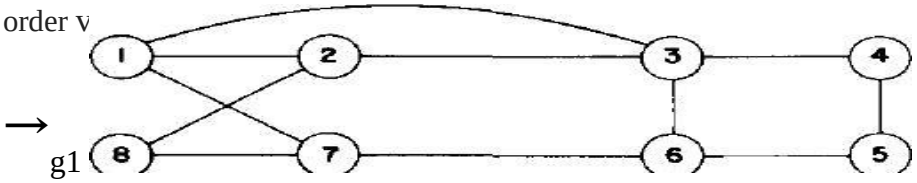
It is also called the Hamiltonian circuit.

Hamiltonian circuit is a graph cycle (i.e., closed loop) through a graph that visits each node exactly once.

A graph possessing a Hamiltonian cycle is said to be Hamiltonian graph. Example:

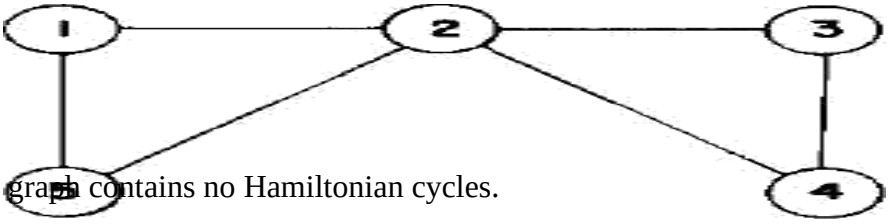


In graph G, Hamiltonian cycle begins at some vertiex v1 $\in$ G and the vertices of G are visited in the order v



→ g1 The

above graph contains Hamiltonian cycle: 1,2,8,7,6,5,4,3,1



The above graph contains no Hamiltonian cycles.

There is no known easy way to determine whether a given graph contains a Hamiltonian cycle.

By using backtracking method, it can be possible

Backtracking algorithm, that finds all the Hamiltonian cycles in a graph.

The graph may be directed or undirected. Only distinct cycles are output.

From graph g1 backtracking solution vector= {1, 2, 8, 7, 6, 5, 4, 3, 1}

The backtracking solution vector (x_1, x_2, --- x_n) x_i → i^{th} visited vertex of proposed cycle.

➢ By using backtracking we need to determine how to compute the set of possible vertices for x_k if $x_1, x_2, x_3 \text{---} x_{k-1}$ have already been chosen.

If k=1 then x1 can be any of the n-vertices.

By using "NextValue" algorithm the recursive backtracking scheme to find all Hamiltoman cycles.

This algorithm is started by 1^{st} initializing the adjacency matrix G[1:n, 1:n] then setting x[2:n] to zero & x[1] to 1, and then executing Hamiltonian (2)

Generating Next Vertex	**Finding all Hamiltonian Cycles**

Algorithm NextValue(k)
{
// x[1: k-1] $\rightarrow$ is path of k-1 distinct vertices.
// if x[k]=0, then no vertex has yet been assigned to x[k]
Repeat{
X[k]=(x[k]+1) mod (n+1); //Next vertex
If(x[k]=0) then return;
If(G[x[k-1], x[k]]≠0) then
{
For j:=1 to k-1 do if(x[j]=x[k]) then break;
//Check for distinctness
If(j=k) then //if true , then vertex is distinct
If((k<n) or (k=n) and G[x[n], x[1]]≠0)) Then return ;
}
}
Until (false);
}
	Algorithm Hamiltonian(k)
	{
	Repeat{
	NextValue(k); //assign a legal next value to x[k]
	If(x[k]=0) then return;
	If(k=n) then write(x[1:n]);
	Else Hamiltonian(k+1);
	} until(false)
	}

Branch & Bound

Branch & Bound (B & B) is general algorithm (or Systematic method) for finding optimal solution of various optimization problems, especially in discrete and combinatorial optimization.

➢ The B&B strategy is very similar to backtracking in that a state space tree is used to solve a problem.

➢ The differences are that the B&B method

✓ Does not limit us to any particular way of traversing the tree.

✓ It is used only for optimization problem

✓ It is applicable to a wide variety of discrete combinatorial problem.

➢ B&B is rather general optimization technique that applies where the greedy method & dynamic programming fail.

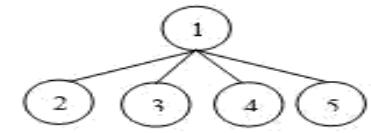
➢ It is much slower, indeed (truly), it often (rapidly) leads to exponential time complexities in the worst case.

➢ The term B&B refers to all state space search methods in which all children of the "E-node" are generated before any other "live node" can become the "E-node"

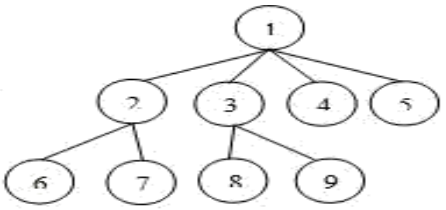
✓ $\rightarrow$ **Live node** is a node that has been generated but whose children have not yet been generated.

✓ $\rightarrow$ **E-node** is a live node whose children are currently being explored.

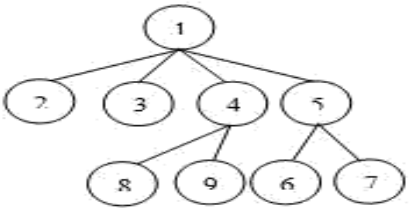
✓ **Dead node** → is a generated node that is not to be expanded or explored any further. All children of a dead node have already been expanded.



Live Node: 2, 3, 4, and 5



FIFO Branch & Bound (BFS)
Children of E-node are inserted in a queue.



LIFO Branch & Bound (D-Search)
Children of E-node are inserted in a stack.

➢ Two graph search strategies, BFS & D-search (DFS) in which the exploration of a new node cannot begin until the node currently being explored is fully explored.

➢ Both BFS & D-search (DFS) generalized to B&B strategies.

✓ **BFS** → like state space search will be called FIFO (First In First Out) search as the list of live nodes is "First-in-first-out" list (or queue).

✓ **D-search (DFS)** → Like state space search will be called LIFO (Last In First Out) search as the list of live nodes is a "last-in-first-out" list (or stack).

➢ In backtracking, bounding function are used to help avoid the generation of sub-trees that do not contain an answer node.

➢ We will use 3-types of search strategies in branch and bound

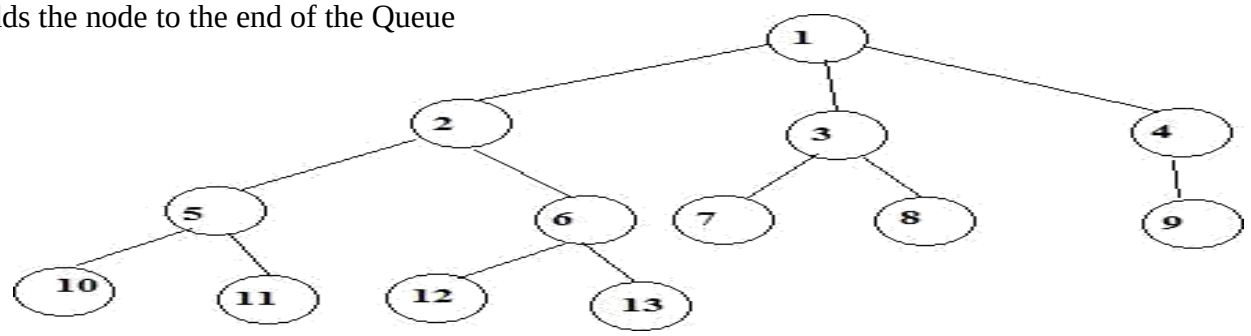
1) FIFO (First In First Out) search
2) LIFO (Last In First Out) search
3) LC (Least Count) search

FIFO B&B:

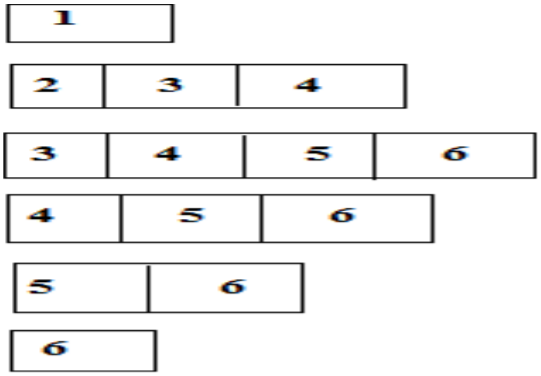
FIFO Branch & Bound is a BFS.
In this, children of E-Node (or Live nodes) are inserted in a queue.
Implementation of list of live nodes as a queue

✓ Least() → Removes the head of the Queue

✓ Add() → Adds the node to the end of the Queue



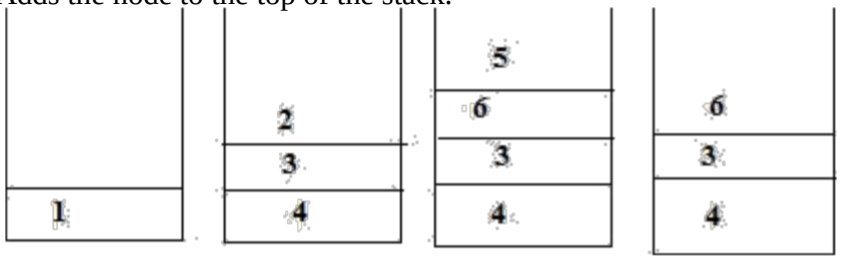
Assume that node '12' is an answer node in FIFO search, 1st we take E-node has '1'



LIFO B&B:

Implementation of List of live nodes as a stack

- Least() → Removes the top of the stack
- ADD() → Adds the node to the top of the stack.



Least Cost (LC) Search:

The selection rule for the next E-node in FIFO or LIFO branch and bound is sometimes "blind". i.e., the selection rule does not give any preference to a node that has a very good chance of getting the search to an answer node quickly.

The search for an answer node can often be speeded by using an "intelligent" ranking function. It is also called an approximate cost function "**Ĉ**".

Expended node (E-node) is the live node with the best **Ĉ** value.

Branching: A set of solutions, which is represented by a node, can be partitioned into mutually (jointly or commonly) exclusive (special) sets. Each subset in the partition is represented by a child of the original node.

Lower bounding: An algorithm is available for calculating a lower bound on the cost of any solution in a given subset.

Each node X in the search tree is associated with a cost: **Ĉ(X)**

C=cost of reaching the current node, X(E-node) form the root + The cost of reaching an answer node form X.

Ĉ=g(X)+H(X).

Example:

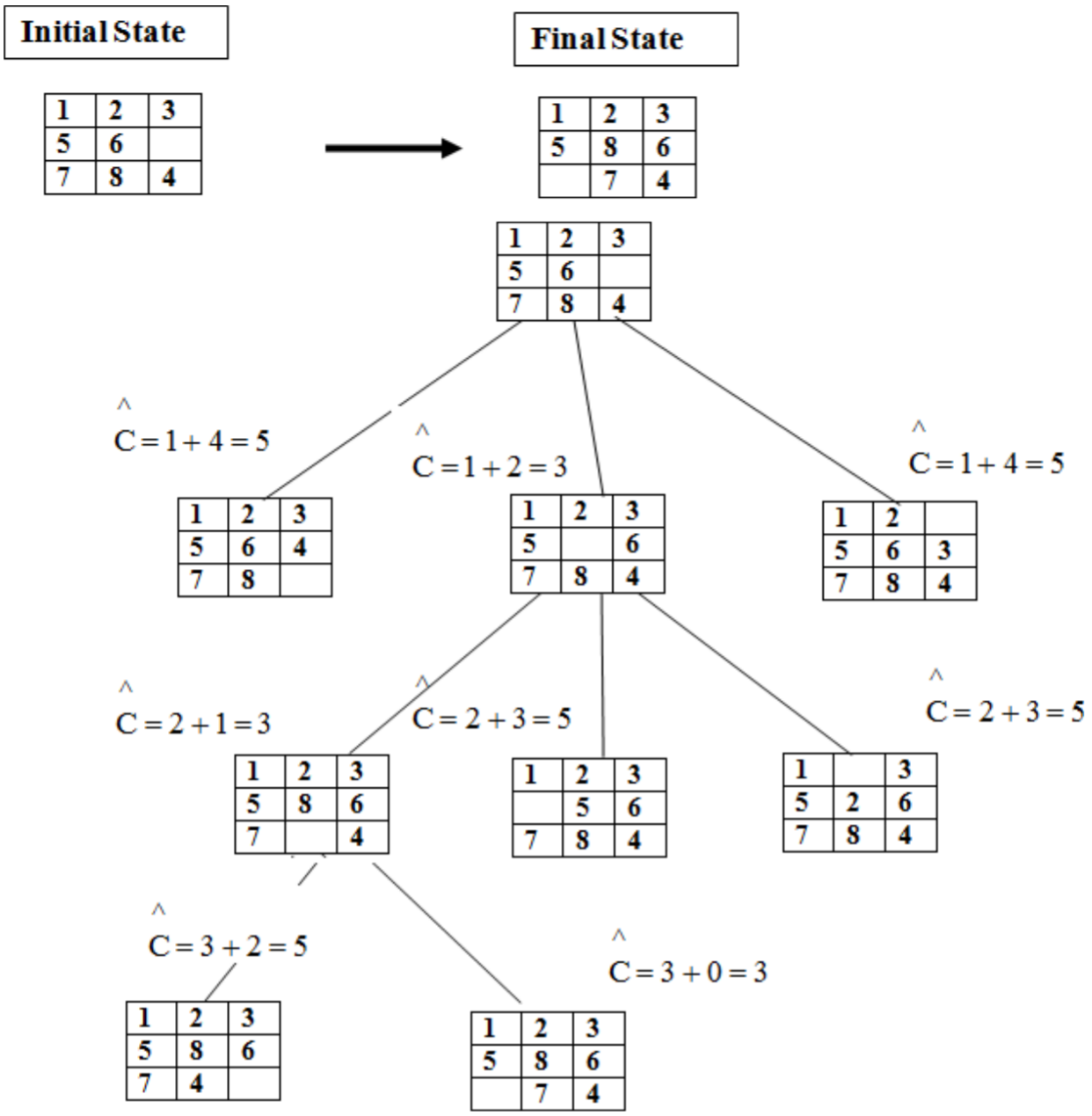
8-puzzle

Cost function: **Ĉ** = g(x) +h(x)

where h(x) = the number of misplaced tiles

and g(x) = the number of moves so far Assumption:

move one tile in any direction cost 1.



Note: In case of tie, choose the leftmost node.

Travelling Salesman Problem:

Def:- Find a tour of minimum cost starting from a node S going through other nodes only once and returning to the starting point S.

Time Conmlexity of TSP for Dynamic Programming algorithm is $O(n^2 2^n)$

B&B algorithms for this problem, the worest case complexity will not be any better than $O(n^2 2^n)$ but good bunding functions will enables these B&B algorithms to solve some problem instances in much less time than required by the dynamic programming alogrithm.

Let G=(V,E) be a directed graph defining an instances of TSP. Let

C_{ij} $\rightarrow$ cost of edge <i, j>

$C_{ij} = \infty$ if <i, j> $\notin E$

$|V|=n$ $\rightarrow$ total number of vertices.

Assume that every tour starts & ends at vertex 1.

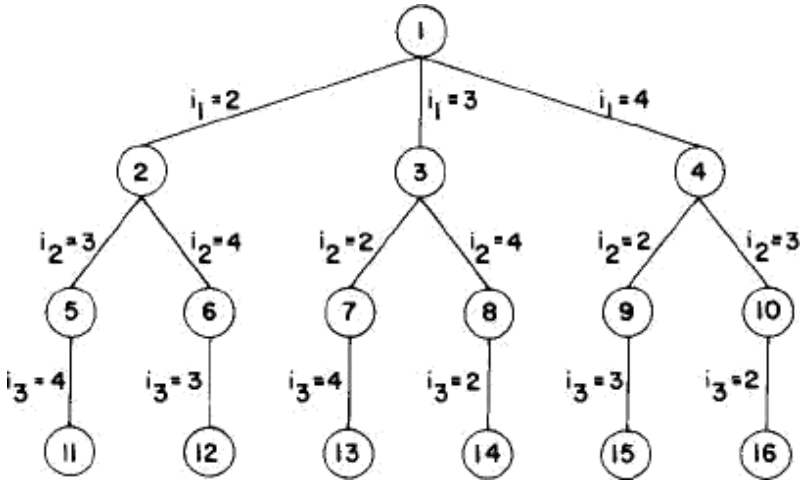
Solution Space S= {1, Π , 1 / Π is a permutation of (2, 3. 4 ---------- n) } then |S|=(n-1)!

The size of S reduced by restricting S

Sothat (1, i_1,i2,-----i_{n-1}, 1}$\in$ S iff <i_j, i_{j+1}>$\in$ E. O$\leq$j$\leq$n-1, i_0-i_n=1

S can be organized into "State space tree".

Consider the following Example



State space tree for the travelling salesperson problem with n=4 and $i_0=i_4=1$ The

above diagram shows tree organization of a complete graph with |V|=4.

Each leaf node 'L' is a solution node and represents the tour defined by the path from the root to L.

Node 12 represents the tour.

$i_0=1, i_1=2, i_2=4, i_3=3, i_4=1$

Node 14 represents the tour.

$i_0=1, i_1=3, i_2=4, i_3=2, i_4=1.$

TSP is solved by using LC Branch & Bound:

To use LCBB to search the travelling salesperson "State space tree" first define a cost function C(.) and other 2 functions $\hat{C}$(.) & u(.)

Such that $\hat{C}(r) \leq C(r) \leq u(r)$ $\rightarrow$ for all nodes r.

Cost C(.) $\rightarrow$ is the solution node1 with least C(.) corresponds to a shortest tour in G.

C(A)={Length of tour defined by the path from root to A if A is leaf Cost of a minimum-cost leaf in the sub-tree A, if A is not leaf }

From1 $\hat{C}(r) \le C(r)$ then $\hat{C}(r) \rightarrow$ is the length of the path defined at node A.

From previous example the path defined at node 6 is i_0, i_1, i_2=1, 2, 4 & it consists edge of <1,2> & <2,4>

Abetter C(r) can be obtained by using the reduced cost matrix corresponding to G.

A row (column) is said to be reduced iff it contains at least one zero & remaining entries are non negative.

A matrix is reduced iff every row & column is reduced.

$$\begin{bmatrix} \infty & 20 & 30 & 10 & 11 \\ 15 & \infty & 16 & 4 & 2 \\ 3 & 5 & \infty & 2 & 4 \\ 19 & 6 & 18 & \infty & 3 \\ 16 & 4 & 7 & 16 & \infty \end{bmatrix}$$

(a) Cost Matrix

$$\begin{bmatrix} \infty & 10 & 17 & 0 & 1 \\ 12 & \infty & 11 & 2 & 0 \\ 0 & 3 & \infty & 0 & 2 \\ 15 & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & 12 & \infty \end{bmatrix}$$

(b) Reduced Cost Matrix

L = 25

Given the following cost matrix:

The TSP starts from node 1: **Node 1**

Reduced Matrix: To get the lower bound of the path starting at node 1

Row # 1: reduce by 10	Row #2: reduce 2	Row #3: reduce by 2

Row # 4: Reduce by 3:	Row # 5: Reduce by 4	Column 1: Reduce by 1

Column 2: It is reduced.	Column 3: Reduce by 3 $\begin{bmatrix} inf & 10 & 17 & 0 & 1 \\ 12 & inf & 11 & 2 & 0 \\ 0 & 3 & inf & 0 & 2 \\ 15 & 3 & 12 & inf & 0 \\ 11 & 0 & 0 & 12 & inf \end{bmatrix}$	Column 4: It is reduced. Column 5: It is reduced.

The reduced cost is: RCL = 25
So the cost of node 1 is: Cost (1) = 25 The reduced matrix is:

Cost (1) = 25

$$\begin{bmatrix} inf & 10 & 17 & 0 & 1 \\ 12 & inf & 11 & 2 & 0 \\ 0 & 3 & inf & 0 & 2 \\ 15 & 3 & 12 & inf & 0 \\ 11 & 0 & 0 & 12 & inf \end{bmatrix}$$

Choose to go to vertex 2: Node 2

- Cost of edge <1,2> is: A(1,2) = 10
- Set row #1 = inf since we are choosing edge <1,2>
- Set column # 2 = inf since we are choosing edge <1,2>
- Set A(2,1) = inf
- The resulting cost matrix is:

- The matrix is reduced:
- RCL=0
- The cost of node 2 (Considering vertex 2 from vertex 1) is:

Cost(2) = cost(1) + A(1,2) = 25 + 10 = 35

➢ **Choose to go to vertex 3: Node 3**

- Cost of edge <1,3> is: A(1,3) = 17 (In the reduced matrix
- Set row #1 = inf since we are starting from node 1
- Set column # 3 = inf since we are choosing edge <1,3>
- Set A(3,1) = inf
- The resulting cost matrix is:

Reduce the matrix: Rows are reduced

The columns are reduced except for column # 1:
Reduce column 1 by 11:

The lower bound is: RCL = 11
The cost of going through node 3 is:
cost(3) = cost(1) + RCL + A(1,3) = 25 + 11 + 17 = 53

➢ Choose to go to vertex 4: **Node 4**

Remember that the cost matrix is the one that was reduced at the starting vertex 1
Cost of edge <1,4> is: A(1,4) = 0
Set row #1 = inf since we are starting from node 1
Set column # 4 = inf since we are choosing edge
<1,4> Set A(4,1) = inf
The resulting cost matrix is:

Reduce the matrix: Rows are reduced

Columns are reduced
The lower bound is: RCL = 0
The cost of going through node 4 is:
cost(4) = cost(1) + RCL + A(1,4) = 25 + 0 + 0 = 25

Choose to go to vertex 5: Node 5

- Remember that the cost matrix is the one that was reduced at starting vertex 1
- Cost of edge <1,5> is: A(1,5) = 1
- Set row #1 = inf since we are starting from node 1
- Set column # 5 = inf since we are choosing edge <1,5>
- Set A(5,1) = inf
- The resulting cost matrix is:

Reduce the matrix:
Reduce rows:
Reduce row #2: Reduce by 2

Reduce row #4: Reduce by 3

Columns are reduced
The lower bound is: RCL = 2 + 3 = 5 The cost of
going through node 5 is:
cost(5) = cost(1) + RCL + A(1,5) = 25 + 5 + 1 = 31

In summary:
So the live nodes we have so far are:

✓ 2: cost(2) = 35, path: 1->2

✓ 3: cost(3) = 53, path: 1->3

✓ 4: cost(4) = 25, path: 1->4

✓ 5: cost(5) = 31, path: 1->5

Explore the node with the lowest cost: Node 4 has a cost of 25
Vertices to be explored from node 4: 2, 3, and 5
Now we are starting from the cost matrix at node 4 is:

Cost (4) = 25

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ 12 & inf & 11 & inf & 0 \\ 0 & 3 & inf & inf & 2 \\ inf & 3 & 12 & inf & 0 \\ 11 & 0 & 0 & inf & inf \end{bmatrix}$$

➢ **Choose to go to vertex 2: Node 6 (path is 1->4->2)** -

Cost of edge <4,2> is: A(4,2) = 3
Set row #4 = inf since we are considering edge <4,2>
Set column # 2 = inf since we are considering edge <4,2>
Set A(2,1) = inf
The resulting cost matrix is:

Reduce the matrix: Rows are reduced
Columns are reduced
The lower bound is: RCL = 0
The cost of going through node 2 is:
cost(6) = cost(4) + RCL + A(4,2) = 25 + 0 + 3 = 28

Choose to go to vertex 3: Node 7 (path is 1->4->3)

Cost of edge <4,3> is: A(4,3) = 12
Set row #4 = inf since we are considering edge <4,3>
Set column # 3 = inf since we are considering edge <4,3>
Set A(3,1) = inf
The resulting cost matrix is:

Reduce the matrix:

Reduce row #3: by 2:

Reduce column # 1: by 11

The lower bound is: RCL = 13
So the RCL of node 7 (Considering vertex 3 from vertex 4) is:
Cost(7) = cost(4) + RCL + A(4,3) = 25 + 13 + 12 = 50

Choose to go to vertex 5: **Node 8** (path is 1->4->5) Cost of

edge <4,5> is: A(4,5) = 0
Set row #4 = inf since we are considering edge <4,5>
Set column # 5 = inf since we are considering edge <4,5> Set A(5,1) = inf
The resulting cost matrix is:

Reduce the matrix: Reduced row 2: by 11

Columns are reduced
The lower bound is: RCL = 11
So the cost of node 8 (Considering vertex 5 from vertex 4) is: Cost(8) = cost(4) + RCL + A(4,5) = 25 + 11 + 0 = 36

In summary: So the live nodes we have so far are:

- ✓ 2: cost(2) = 35, path: 1->2
- ✓ 3: cost(3) = 53, path: 1->3
- ✓ 5: cost(5) = 31, path: 1->5
- ✓ 6: cost(6) = 28, path: 1->4->2
- ✓ 7: cost(7) = 50, path: 1->4->3
- ✓ 8: cost(8) = 36, path: 1->4->5

➢ Explore the node with the lowest cost: Node 6 has a cost of 28
Vertices to be explored from node 6: 3 and 5

➢ Now we are starting from the cost matrix at node 6 is:

Cost (6) = 28

$$\begin{bmatrix} inf & inf & inf & inf & inf \\ inf & inf & 11 & inf & 0 \\ 0 & inf & inf & inf & 2 \\ inf & inf & inf & inf & inf \\ 11 & inf & 0 & inf & inf \end{bmatrix}$$

Choose to go to vertex 3: Node 9 (path is 1->4->2->3)

Cost of edge <2,3> is: A(2,3) = 11
Set row #2 = inf since we are considering edge <2,3>
Set column # 3 = inf since we are considering edge <2,3>
Set A(3,1) = inf
The resulting cost matrix is:

Reduce the matrix: Reduce row #3: by 2

Reduce column # 1: by 11

The lower bound is: RCL = 2 +11 = 13
So the cost of node 9 (Considering vertex 3 from vertex 2) is:
Cost(9) = cost(6) + RCL + A(2,3) = 28 + 13 + 11 = 52

Choose to go to vertex 5: Node 10 (path is 1->4->2->5)

Cost of edge <2,5> is: A(2,5) = 0
Set row #2 = inf since we are considering edge <2,3>
Set column # 3 = inf since we are considering edge <2,3>
Set A(5,1) = inf
The resulting cost matrix is:

Reduce the matrix: Rows reduced
Columns reduced
The lower bound is: RCL = 0
So the cost of node 10 (Considering vertex 5 from vertex 2) is:
Cost(10) = cost(6) + RCL + A(2,3) = 28 + 0 + 0 = 28

In summary: **So the live nodes we have so far are:**

- ✓ 2: cost(2) = 35, path: 1->2
- ✓ 3: cost(3) = 53, path: 1->3
- ✓ 5: cost(5) = 31, path: 1->5
- ✓ 7: cost(7) = 50, path: 1->4->3
- ✓ 8: cost(8) = 36, path: 1->4->5
- ✓ 9: cost(9) = 52, path: 1->4->2->3
- ✓ 10: cost(2) = 28, path: 1->4->2->5

➢ Explore the node with the lowest cost: Node 10 has a cost of 28
Vertices to be explored from node 10: 3

Now we are starting from the cost matrix at node 10 is:

Cost (10)=28

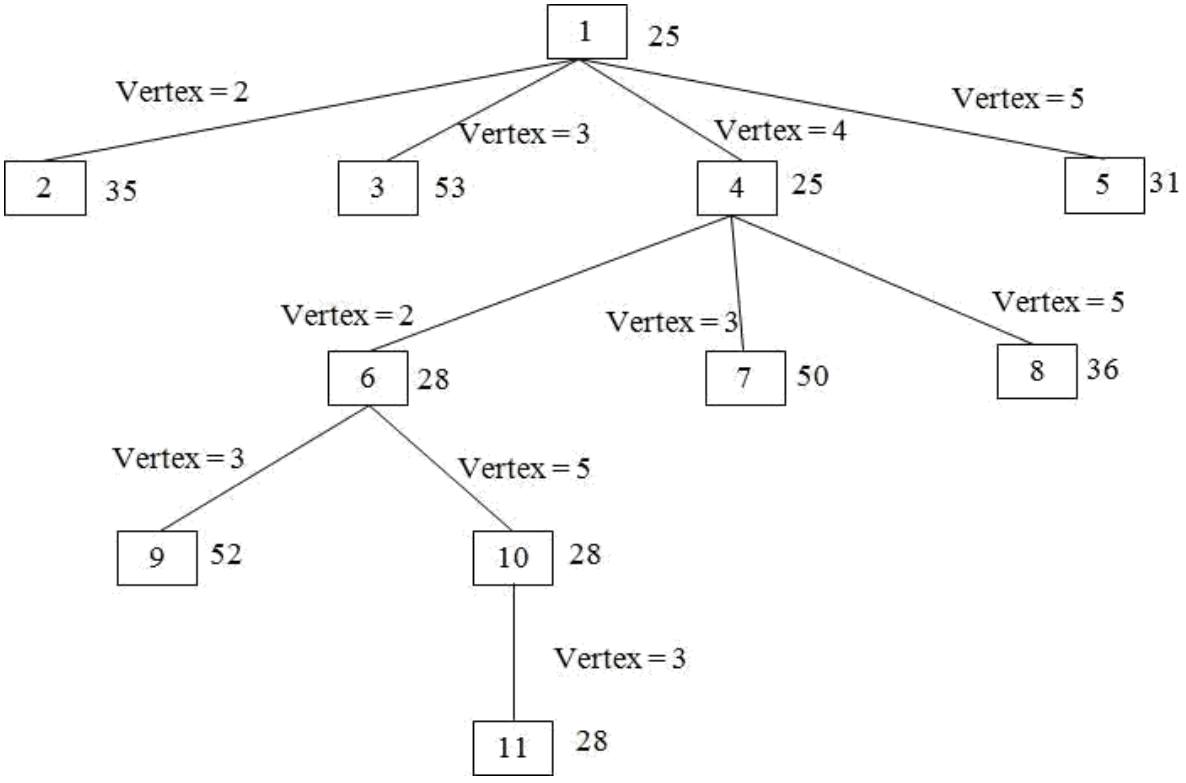
$$\begin{bmatrix} inf & inf & inf & inf & inf \\ inf & inf & inf & inf & inf \\ 0 & inf & inf & inf & inf \\ inf & inf & inf & inf & inf \\ inf & inf & 0 & inf & inf \end{bmatrix}$$

Choose to go to vertex 3: Node 11 (path is 1->4->2->5->3)

Cost of edge <5,3> is: A(5,3) = 0
Set row #5 = inf since we are considering edge <5,3>
Set column # 3 = inf since we are considering edge <5,3>
Set A(3,1) = inf
The resulting cost matrix is:

Reduce the matrix: Rows reduced
Columns reduced
The lower bound is: RCL = 0
So the cost of node 11 (Considering vertex 5 from vertex 3) is:
Cost(11) = cost(10) + RCL + A(5,3) = 28 + 0 + 0 = 28

State Space Tree:



O/1 Knapsack Problem

What is Knapsack Problem: Knapsack problem is a problem in combinatorial optimization, Given a set of items, each with a mass & a value, determine the number of each item to include in a collection so that the total weight is less than or equal to a given limit & the total value is as large as possible.
O-1 Knapsack Problem can formulate as. Let there be n items, Z_1 **to** Z_n where Z_i has value P_i & weight w_i. The maximum weight that can carry in the bag is m. All values and weights are non negative.
Maximize the sum of the values of the items in the knapsack, so that sum of the weights must be less than the knapsack's capacity m.
The formula can be stated as

$$\text{maximize} \sum_{1 \le i \le n} p_i x_i$$

$$\text{subject to} \sum_{1 \le i \le n} w_i x_i \le M$$

X_i=0 or 1 1 ≤ i ≤ n

To solve o/1 knapsack problem using B&B:

- Knapsack is a maximization problem
- Replace the objective function $\sum p_i x_i$ by the function $-\sum p_i x_i$ to make it into a minimization problem

The modified knapsack problem is stated as

$$\text{Minimize} -\sum_{i=1}^{n} p_i x_i$$

$$\text{subject to} \sum_{i=1}^{n} w_i x_i < m,$$

$$x_i \in \{0,1\}, 1 \le i \le n$$

Fixed tuple size solution space:

- Every leaf node in state space tree represents an answer for which $\sum_{1 \le i \le n} w_i x_i \leqslant m$ is an answer node; other leaf nodes are infeasible
- For optimal solution, define

$$c(x) = -\sum_{1 \le i \le n} p_i x_i$$

for every answer node x.

- For infeasible leaf nodes, $c(x) = \infty$
- For non leaf nodes

c(x) = min{c(lchild(x)), c(rchild(x))}

- Define two functions cˆ(x) and u(x) such that for every node x,

cˆ(x) ≤ c(x) ≤ u(x)

Computing cˆ(·) and u(·)

Let x be a node at level j, $1 \leq j \leq n+1$

Cost of assignment: $-\sum_{1 \leq i < j} p_i x_i$

$c(x) \leq -\sum_{1 \leq i < j} p_i x_i$

We can use $u(x) = -\sum_{1 \leq i < j} p_i x_i$

Using $q = -\sum_{1 \leq i < j} p_i x_i$, an improved upper bound function $u(x)$ is

$$u(x) = \text{ubound}(q, \sum_{1 \leq i < j} w_i x_i, j-1, m)$$

```
Algorithm ubound ( cp, cw, k, m )
{
// Input:  cp:  Current profit total
// Input:  cw: Current weight total
// Input:  k:  Index of last removed item
// Input:  m:  Knapsack capacity
b=cp; c=cw;
for i:=k+1 to n do{ if(c+w[i] ≤ m) then
{
c:=c+w[i]; b=b-p[i];
}
}
return b;
}
```

MODULE V:

NP-Hard and NP-Complete problems: Basic concepts, non deterministic algorithms, NP - Hard and NPComplete classes, Cook's theorem.

Basic concepts:

NP, Nondeterministic Polynomial time

The problems has best algorithms for their solutions have "Computing times", that cluster into two groups

Group 1	**Group 2**
> Problems with solution time bound by a polynomial of a small degree. > It also called "Tractable Algorithms" > Most Searching & Sorting algorithms are polynomial time algorithms > **Ex:** Ordered Search **(O (log n)),** Polynomial evaluation **O(n)** Sorting **O(n.log n)**	> Problems with solution times not bound by polynomial (simply non polynomial) > These are hard or intractable problems > None of the problems in this group has been solved by any polynomial time algorithm > **Ex:** Traveling Sales Person **$O(n^2 2^n)$** Knapsack **$O(2^{n/2})$**

No one has been able to develop a polynomial time algorithm for any problem in the 2nd group (i.e., group 2)

So, it is compulsory and finding algorithms whose computing times are greater than polynomial very quickly because such vast amounts of time to execute that even moderate size problems cannot be solved.

Theory of NP-Completeness:

Show that may of the problems with no polynomial time algorithms are computational time algorithms are computationally related.

There are two classes of non-polynomial time problems

1. NP-Hard
2. NP-Complete

NP Complete Problem: A problem that is NP-Complete can solved in polynomial time if and only if (iff) all other NP-Complete problems can also be solved in polynomial time.

NP-Hard: Problem can be solved in polynomial time then all NP-Complete problems can be solved in polynomial time.

All NP-Complete problems are NP-Hard but some NP-Hard problems are not know to be NP-Complete.

Nondeterministic Algorithms:

Algorithms with the property that the result of every operation is uniquely defined are termed as deterministic algorithms. Such algorithms agree with the way programs are executed on a computer.

Algorithms which contain operations whose outcomes are not uniquely defined but are limited to specified set of possibilities. Such algorithms are called nondeterministic algorithms.

The machine executing such operations is allowed to choose any one of these outcomes subject to a termination condition to be defined later.

To specify nondeterministic algorithms, there are 3 new functions.

Choice(S), arbitrarily chooses one of the elements of sets S

Failure (), Signals an Unsuccessful completion

Success (), Signals a successful completion.

Example for Non Deterministic algorithms:

Algorithm Search(x){
//Problem is to search an element x
//output J, such that A[J]=x; or J=0 if x is not in A
J:=Choice(1,n);
if(A[J]:=x) then {
Write(J); Success();
}
else{
write(0); failure();
	Whenever there is a set of choices that leads to a successful completion then one such set of choices is always made and the algorithm terminates.
	A Nondeterministic algorithm terminates unsuccessfully if and only if (iff) there exists no set of choices leading to a successful signal.

}	

Nondeterministic Knapsack algorithm	

Algorithm DKP(p, w, n, m, r, x){
W:=0;
P:=0;
for i:=1 to n do{
x[i]:=choice(0, 1);
W:=W+x[i]*w[i];
P:=P+x[i]*p[i];
}
if((W>m) or (P<r)) then Failure();
else Success();
}
	p) given Profits
	w) given Weights
	n) Number of elements (number of p or w)
	m) Weight of bag limit
	P) Final Profit
	W) Final weight

The Classes NP-Hard & NP-Complete:

For measuring the complexity of an algorithm, we use the input length as the parameter. For example, An algorithm A is of polynomial complexity p() such that the computing time of A is O(p(n)) for every input of size n.

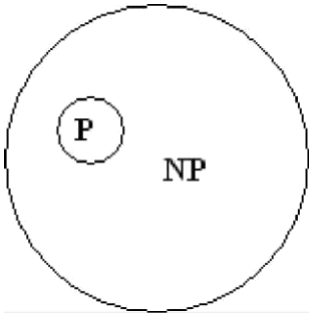
Decision problem/ Decision algorithm: Any problem for which the answer is either zero or one is decision problem. Any algorithm for a decision problem is termed a decision algorithm.

Optimization problem/ Optimization algorithm: Any problem that involves the identification of an optimal (either minimum or maximum) value of a given cost function is known as an optimization problem. An optimization algorithm is used to solve an optimization problem.

P) is the set of all decision problems solvable by deterministic algorithms in polynomial time.

NP) is the set of all decision problems solvable by nondeterministic algorithms in polynomial time.

Since deterministic algorithms are just a special case of nondeterministic, by this we concluded that **P ⊆ NP**



Commonly believed relationship between P & NP

The most famous unsolvable problems in Computer Science is Whether P=NP or P≠NP In considering this problem, s.cook formulated the following question.

If there any single problem in NP, such that if we showed it to be in 'P' then that would imply that P=NP.

Cook answered this question with

Theorem: Satisfiability is in P if and only if (iff) P=NP -

)Notation of Reducibility

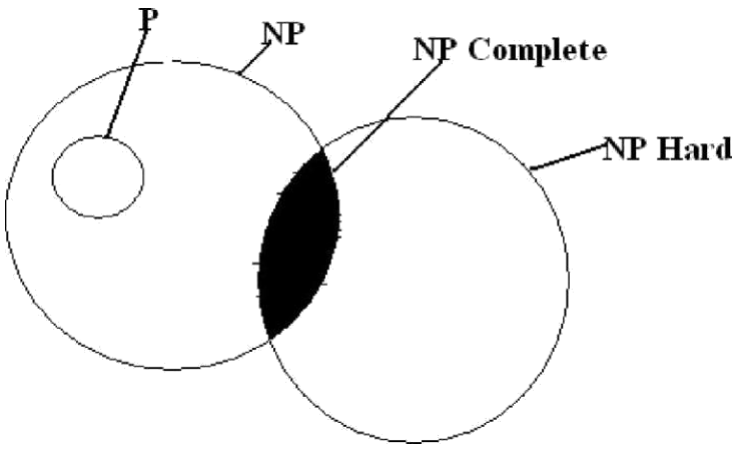
Let L_1 and L_2 be problems, Problem L_1 reduces to L_2 (written $\mathbf{L_1 \alpha L_2}$) iff there is a way to solve L_1 by a deterministic polynomial time algorithm using a deterministic algorithm that solves L_2 in polynomial time

This implies that, if we have a polynomial time algorithm for L_2, Then we can solve L_1 in polynomial time.

Here α-) is a transitive relation i.e., $\mathbf{L_1 \alpha L_2}$ and $\mathbf{L_2 \alpha L_3}$ then $\mathbf{L_1 \alpha L_3}$

A problem L is NP-Hard if and only if (iff) satisfiability reduces to L ie., **Statisfiability α L**

A problem L is NP-Complete if and only if (iff) L is NP-Hard and **L Є NP**



Commonly believed relationship among P, NP, NP-Complete and NP-Hard

Most natural problems in NP are either in P or NP-complete.

Examples of NP-complete problems:

- Packing problems: SET-PACKING, INDEPENDENT-SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3-COLOR, CLIQUE.
- Constraint satisfaction problems: SAT, 3-SAT.
- Numerical problems: SUBSET-SUM, PARTITION, KNAPSACK.

Cook's Theorem: States that satisfiability is in P if and only if P=NP If P=NP then satisfiability is in P
If satisfiability is in P, then P=NP
To do this

> A-) Any polynomial time nondeterministic decision algorithm.

I-)Input of that algorithm

Then formula Q(A, I), Such that Q is satisfiable iff '**A'** has a successful termination with Input **I**.

> If the length of '**I**' is '**n'** and the time complexity of **A** is **p(n)** for some polynomial **p()** then length of Q is $\mathbf{O(p^3(n) \log n)=O(p^4(n))}$

The time needed to construct Q is also $\mathbf{O(p^3(n) \log n).}$

> A deterministic algorithm '**Z'** to determine the outcome of '**A**' on any input '**I**' Algorithm **Z** computes '**Q**' and then uses a deterministic algorithm for the satisfiability problem to determine whether '**Q**' is satisfiable.

> If **O(q(m))** is the time needed to determine whether a formula of length '**m**' is satisfiable then the complexity of '**Z**' is $\mathbf{O(p^3(n) \log n + q(p^3(n) \log n)).}$

> If satisfiability is '**p**', then '**q(m)**' is a polynomial function of '**m**' and the complexity of '**Z**' becomes '**O(r(n))'** for some polynomial '**r()'.**

> Hence, if satisfiability is in **p**, then for every nondeterministic algorithm **A** in **NP**, we can obtain a deterministic **Z** in **p.**

By this we shows that satisfiability is in **p** then **P=NP**